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ABSTRACT

The activities in this monograph are self-contained explorations into mathematical topics not often included in regular classroom work in grades 3-8. They were selected because of their intriguing enjoyment and because they deal with honest mathematics; they also strengthen skills, especially problem-solving skills. Included are activities with geoboards, pentominoes, probability, solid shapes, symmetry, tangrams, tiling, topology, triangles, and dot paper. Introductory comments to the teacher precede each topic; each investigation is presented, then followed by worksheets. (MNS)

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Guidelines for Quality Mathematics Teaching

A Monograph Series

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INVESTIGATIONS
IN
MATHEMATICS

Janet Morris
The University of Michigan - Flint

February, 1979

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Besides providing mathematical background, the "Notes to the Instructor" are intended to explain objectives, to point out difficulties the instructor might not anticipate, suggest alternate procedures and key directive questions, provide some answers, and suggest extensions to the topic. After a general discussion section, the "Notes" contain a section, "Comments: Specific", which are numbered to correspond to the specific parts to which they relate, first to the Investigation, and then to the Worksheet.

INTRODUCTION

Investigations in Mathematics are self-contained explorations into topics of mathematics which are not often included in the regular classroom work, but which are, nevertheless, well within the grasp of students in grades 3-8. Two main goals were considered in choosing these topics: that they were intriguing "fun", and that they dealt with honest mathematics. The most appropriate work for the faster, interested student is not necessarily that of the next grade level. There are many topics which are available to young explorers which can strengthen their skills (particularly problem solving skills) while giving them a broader idea of what and where mathematics is. While working with Möbius Bands (see "Topology") or coloring maps (see "Solid Shapes") some children will comment, "This isn't like math at all!" But it is mathematics! The children are learning that mathematics is much more than computations.

Doing mathematics, whether as an elementary-middle school student or as a professional, involves trying things, noting outcomes, searching for patterns, making predictions, and then checking their accuracy. It is this spirit which these Investigations hope to capture: the creative process of raising and trying to answer questions, first by trial-and-error, later by using strategies and making generalizations. Doing mathematics on the elementary-middle school level often involves using materials. Even mature students understand more if they examine concrete examples. Thus each Investigation involves materials which the children actively manipulate. The Worksheets attempt to reiterate what the work with the materials revealed, and extend and apply it. Although some Worksheets may not specifically call for concrete materials, the teacher may wish to have the children use them to clarify a point.

The ideas dealt with through the materials of the Investigations are honest mathematics. This is apparent as the students consider famous problems of mathematics (e.g., the Königsberg Bridge Problem, the Four Color Map Problem) and works of some famous mathematicians (e.g., Euler's Formula).

Although each Investigation is self-contained, several have linking ideas which demonstrate the inter-relatedness of mathematical topics. For example, Euler's Formula developed in "Solid Shapes" can be applied to the networks in "Topology". There is no suggested order to the Investigations since they are independent, and each contains activities which learners may deal with in a variety of ways from trial and error to using specific problem solving strategies.

It is suggested that each student keep a notebook in which he/she records his/her observations. After recording his/her results for the Investigation, he/she then has a place in which to add observations he/she makes outside the classroom. For example, after being introduced to symmetry, children enjoy collecting real world instances they find. Such activities reinforce one of the objectives of these Investigations: developing an awareness that mathematics is everywhere: in art, in maps of geography, even in the symmetry of their own bodies, and not just in the textbook.

Each Investigation cites references used to compile the activities of the Investigation and the Worksheets. Although they are not intended to be exhaustive, they will provide the instructor with any further background needed beyond that which is in the "Notes".

GEOBOARDS

Notes to the Instructor

Objectives:
- to create and investigate geometric shapes on 25 peg geoboards;
- to reinforce the concept of area as the number of units needed to cover a region;
- to find the area of a right triangle as ½ the area of the rectangle with base and height the legs of the triangle;
- to find the area of a region by partitioning it into rectangles and right triangles, and counting the units in these;
- to use pattern findings to formulate a rule and to test it, for the area of a simple polygon on the geoboard given the number of pegs on the boundary and the number of pegs inside the polygon.

Materials:
- Geoboards: 5 x 5 peg boards, one per student.
- Rubber bands: assorted colors and sizes, 12 - 14 per student.
- Dot paper: one sheet per student.
- Pencils: one per student.
- Worksheet: one per student.

References:

- Members of the Association of Teachers of Mathematics. Notes on Geoboards. New Rochelle, N.Y., Cuisenaire Company of America, 1967.
- Aman, "Discovery on a Geoboard." Arithmetic Teacher (April 1974): 267.
- Liedtke. "Geoboard Mathematics." Arithmetic Teacher (April 1974): 273.
- Wells. "Creating Mathematics with a Geoboard." Arithmetic Teacher (April 1970): 47.

Comments: General

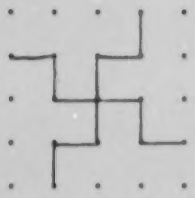
The possibilities for using geoboards are almost as diverse as the children using them. The questions given in the Investigation may give way to questions that develop as the children discover many of the things that can be done on a geoboard. The instructor should feel free to pursue any of these questions that seem more suitable than the suggested work with area. The enormous flexibility and adaptability of work with geoboards makes it one of the more popular classroom aids.

The basic concept of area is developed by counting the number of unit squares it takes to cover the region rather than by using formulas. By establishing this basis of the concept of area, later work with formulas will be more meaningful.

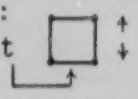
Because area and perimeter are often confused, it is suggested that this Investigation be limited to finding areas. The number of inches or unit segments between pegs is not referred to since it is not needed to find the areas. Students who have a firm grasp of the concepts of area and perimeter may choose to investigate both on the geoboard regions.

This Investigation only begins the possibilities with the geoboard. Other possibilities include the following.

1. Fractions: Can you divide your geoboard into fourths using two rubber bands? How many ways can you find to do this? Did you find this way?



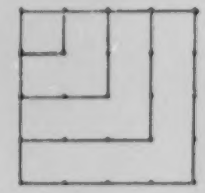
2. Lengths of line segments: Define the unit length as the length of the side of the unit square:



How many segments of a given length (eq. 3 units) can you make on the geoboard? How many different length segments can you make? (You will need to use the Pythagorean Theorem to find the lengths of diagonal segments.)

3. Comparing area and perimeter: (a) Make a triangle and a square having equal areas. Which has the smaller perimeter? Try it again with a different triangle and square. Will a triangle always have a greater perimeter than a square of the same area? (b) Make two non-congruent rectangles each with area 4. Which has a larger perimeter? Make three non-congruent triangles each with area 2. Are their perimeters equal? Do figures with equal areas necessarily have equal perimeters?

4. Number patterns: Find the areas of each of the four squares. What sequence of numbers is formed? Find the area in each of the four non-overlapping L-shaped regions. What sequence of numbers is formed? Can the numbers in the second sequence be summed in a certain way to form the numbers in the first sequence?



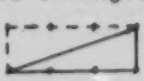
5. Transformations: With a single movement (looping over a peg or releasing the band from a peg) can you make a parallelogram into a triangle? Into a pentagon? Into a rectangle? Can you make a triangle into a square, then into a rhombus, then into a parallelogram, then into a quadrilateral? Notice when you can change a figure into a different figure and why you can; when you cannot and why you cannot. Develop some more complicated movements--looping over two pegs at once, or performing a loop followed by a release, for example--and study the effect of these.

6. Game: Find My Polygon
   Number of Players: two or more.
   Materials: Geoboard and rubber bands for each player.
   Procedure: One player is selected as leader, and constructs a polygon using a single rubber band on his board. He does not allow the other players to see his polygon. The players then take turns asking the leader questions about his polygon that can be answered by a "yes" or "no". The first player to construct the leader's polygon on his board wins, and becomes the next leader.

Comments: Specific for the Investigation

1. If this is the first time the students have worked with the geoboard, a period of time must be allowed during which they find out how different shapes can be made with the rubber bands on the pegs. They should find that shapes with straight sides, such as triangles, rectangles, hexagons, etc., can be made, but those with curved parts such as circles, cannot be made.

2. It is assumed that the students have had experiences with measuring, and understand the idea of a unit of measure: that the choice of the unit is arbitrary but, once chosen, most be consistently used. Point out that the diagonal of the rectangle partitions the rectangle into two right triangles. Since they are the same size, the area of each is ½ the area of the rectangle, or ½ of 2 = 1.

3. Emphasize the strategy of making right triangles into rectangles whose areas can be counted. For example:  so the triangle is one-half the area of the rectangle, ½ of 3 or 1½.

4. Allow plenty of time for this. Many students will readily find 8 to 10 shapes with area 4. Bring out the observation that different regions can have the same area.

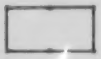
GEOBOARDS

Investigation

1. Find out what shapes you can make on the geoboard with the rubber bands. Can you make every shape whose name you know? Can you make a shape whose name you do not know? A shape you make with one rubber band that does not cross itself is a polygon; the shape has straight sides.

2. We will say that the square with one nail at each corner and no pegs inside has an area of one square unit:

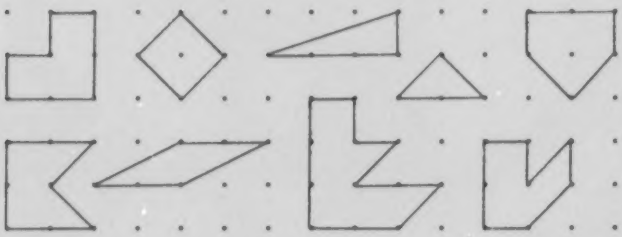
Make this shape on your board:



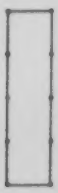
What is its area?
Is it made from two square units?
Where would you put a rubber band
to show it is made from two square units?
Now put a rubber band from one corner peg to the other corner.
It is along the diagonal of the rectangle.
What shapes are formed?
Tell everything you can about these shapes.
What is the area of each?

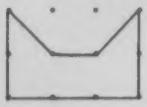
What is the area of this triangle?
How do you know?

3. Make each of these shapes on your board, and find its area. Use extra rubber bands to divide the shape into right triangles and rectangles whose areas you know.



4. Make each of these shapes on your geoboard. What is the area of each? How many more shapes can you make with this same area?





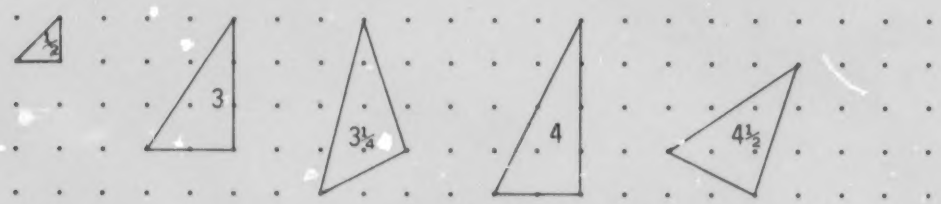
5. The largest area is the square of area 16. Next largest is formed by subtracting ½ from this square by releasing the rubber band from one corner peg to form a 5-sided figure of area 15½.

6. If more than one rubber band is allowed in the constructions, some complex shapes can be made. However, the basic idea of right triangles having area ½ that of the corresponding rectangle is brought out more clearly restricting constructions to one rubber band.

7. Emphasize that only a right triangle has area equal to one-half the area of the corresponding rectangle since it is only right triangles which are formed by a diagonal of a rectangle.

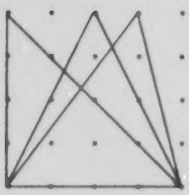
   a) areas left to right are 1, 2, 1, 1½, 2½.

   b)



(Several solutions possible for each)

8.



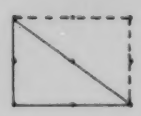
Since they have the same base, and equal heights, the triangle's areas are equal. Have the faster students find and compare the perimeters of the triangles to notice that triangles with equal areas may have different perimeters.

9. This is a lengthy exercise, and you may want to leave it for the students to finish on their own. It may take a few directed questions to get the students to start finding "tilted squares", (those whose sides are not parallel to the edges of the geoboard). Squares of areas listed are possible; areas 3, 6, 7, 11, 13, 14, 15 are not possible on a 25 peg board. Squares with fractional areas can also be found, especially if constructions with more than one rubber band are allowed so vertices need not be on pegs. Squares of areas 4/5, 3 1/5, and 3 3/5 are not difficult to find.

5. Make the largest square you can on the geoboard. What is its area? What is the largest shape you can make that is not a square? What is its area?

6. Make the smallest square you can with one rubber band. Can you make a triangle whose area is one half the area of this square? Is this the smallest shape you can make on the geoboard?

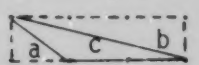
7. You have been finding the area of a right triangle by forming the rectangle of which the triangle is one-half:



Area of rectangle: 4

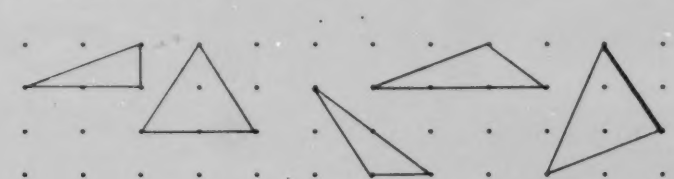
Area of triangle: ½ of 4 = 2

Sometimes we want to find the area of triangles that are not right triangles and so cannot be one-half of a rectangle. In this case, make a big rectangle and subtract the areas of the right triangles around the outside:



Area of rectangle: 3
Area of right triangle "a": ½
Area of right triangle "b": 1½
Area of triangle "c": 3 - ½ - 1½ = 1

a) Make each of these triangles on your geoboard and find the area of each.



b) Make triangles with these areas: ½, 3, 3½, 4, 4½.

8. Make a triangle with area 8 square units. Is this the largest triangle you can make on your geoboard? How do you know? Can you make three different triangles of area four: one right, one isosceles and one scalene? What is the same about each of these? What is different?

9. Make as many different sized squares as you can on your board using a single rubber band. Record the size of each before you remove the rubber band that forms it.
How many did you find?
Did you find squares with these areas: 1, 2, 4, 5, 8, 9, 10, 16?
Did you find any others?

10. a) All shapes with 4 pegs on the boundary and none inside have area 1.

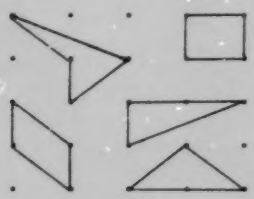
b) Shapes with 5 pegs on the boundary and none inside have area 1½. Draw out that number of pegs on the boundary and inside determines the area of the shape.

c) Emphasize the pattern in the chart to lead to the discovery that $A = \frac{T}{2} + I - 1$. This is known as Pick's Formula. Using Pick's Formula on the given figure: $A = \frac{7}{2} + 6 - 1 = 8\frac{1}{2}$. This formula leads to some interesting questions. It shows that any region formed by a single rubber band that doesn't cross itself must have an area of a whole number or of a whole number plus ½. What other rational number areas are possible? Is it possible to construct a region whose area is not a rational number?

I: number of pegs inside

T: number of pegs touched	0	1	2	3	4
3	½	1½	2½	3½	4½
4	1	2	3	4	5
5	1½	2½	3½	4½	5½
6	2	3	4	5	6
7	2½	3½	4½	5½	6½
8	3	4	5	6	7

10. a) What is the area of each of these shapes?
Do all the shapes with 4 pegs on the boundary (touching the rubber band) and no pegs inside have the same area? Try some more shapes and see.



b) Do all the shapes with 5 pegs on the boundary and no pegs inside have the same area? Try some and see.
What about shapes with 6 pegs on the boundary and 2 inside?
What pattern do you see?

c) FOR EXPERTS ONLY: Copy this table on a piece of paper.
Make the shapes described by their pegs, and find their areas.
T is the number of pegs on the boundary, I is the number of pegs inside the shape, and A is the area of the shape.

I: number of pegs inside

T: number of pegs touched	0	1	2	3	4
3	½	1½	2½	3½	4½
4	1	2	3	4	5
5	1½	2½	3½	4½	5½
6	2	3	4	5	6
7	2½	3½	4½	5½	6½
8	3	4	5	6	7

Study the table. Can you see how to get the area A from T and I?

Hint: Is $\frac{T}{2} + I$ always greater than A?

How is A related to T and I?
Test your answer on this figure.

GEOBOARD

Worksheet

1. On the geoboard, you made shapes and found their areas by counting how many unit squares it would take to cover the shape. Use the same ideas to find the areas of these shapes. Draw in lines that help you find the total area by making the shape into squares, rectangles, and right triangles. Some have been started for you.

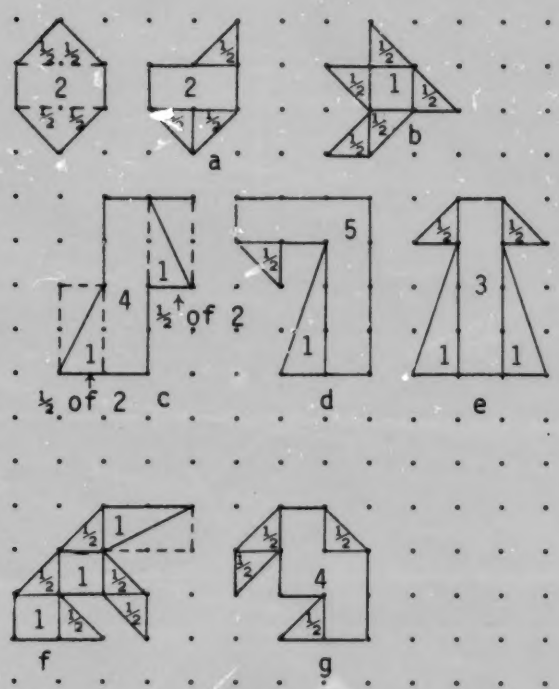
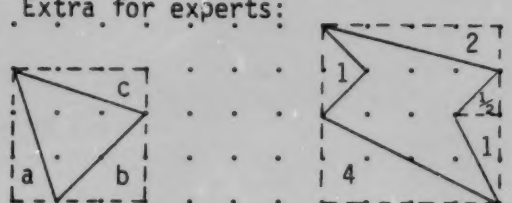


FIGURE	TOTAL AREA
a	3½
b	3½
c	6
d	6½
e	6
f	5½
g	6
Extra I	4
Extra II	7½

Now you make one!

Extra for experts:



16-4-1-2-½-1=7½

For I: Area of rectangle = 9

Area of right triangle = 1½

Area of right triangle b = 2

Area of right triangle c = 1½

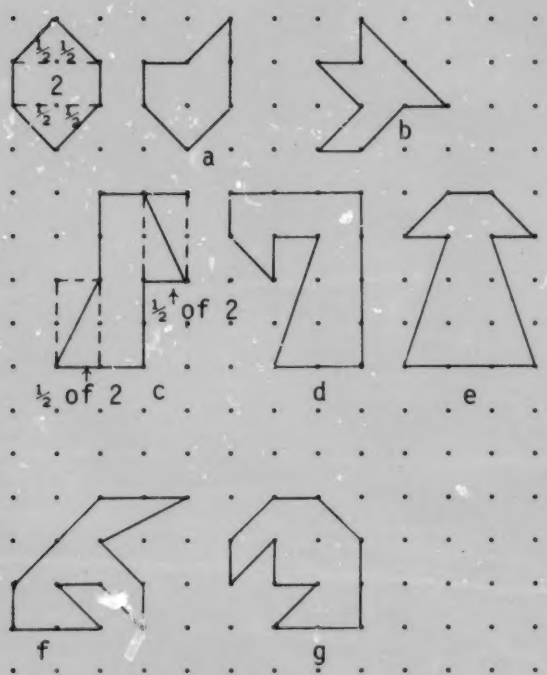
So area of I = 4

GEOBOARD

Worksheet

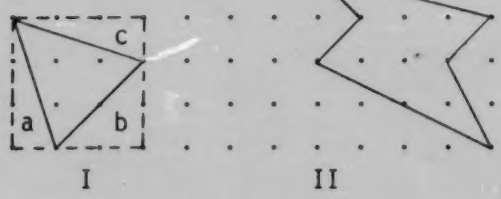
1. On the geoboard, you made shapes and found their areas by counting how many unit squares it would take to cover the shape. Use the same ideas to find the areas of these shapes. Draw in lines that help you find the total area by making the shape into squares, rectangles, and right triangles. Some have been started for you.

FIGURE	TOTAL AREA
a	
b	
c	
d	
e	
f	
g	
Extra I	
Extra II	



Now you make one!

Extra for experts:



For I: Area of rectangle = _____

Area of right triangle = _____

Area of right triangle b = _____

Area of right triangle c = _____

So area of I = _____

PENTOMINOES

Notes to the Instructor

Objectives: to search for patterns and relationships among arrangements of squares, and to develop strategies for finding all such arrangements;
to practice representing a configuration on a grid;
to visualize forming a 3-dimensional shape from a 2-dimensional shape;
to recognize congruent shapes in different orientations;
to find that shapes with equal areas may have different perimeters;
to determine if a shape has line or rotational symmetry;
to find that increasing the dimensions of a polygon by a factor of 3 increases the area by a factor of 9.

Materials: squares: 1" cutouts, 5 per student.
grid paper: 1" grid, 2-3 sheets per student.
scissors: 1 pair per student.
game board: 8" square playing board marked off in 1" grid.
pencils: 1 per student.

References:
Ferrell, Phyllis. "A Pentomino Unit" Mathematics Teacher (September 1977): 523-527.
Gardner, Martin. The Scientific American Book of Mathematical Puzzles and Diversions, New York: Simon and Schuster, 1959: 124-140.
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Walter, Marion. "A Second Example of Informal Geometry: Milk Cartons." Arithmetic Teacher. (May 1969): 368-370.

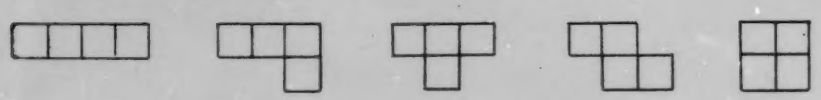
Comments: General

Work with pentominoes is adaptable to a wide range of student ability and maturity. Even younger children become intrigued with the challenge of finding the different possible shapes, enjoy searching for them on a trial-and-error level, and gain valuable experience in spatial visualization. The older children should be encouraged to look for patterns and relationships, and develop strategies for systematically searching for different possibilities. Also, they should mentally visualize and predict the outcome of a physical manipulation such as moving or folding a geometrical figure, before physically performing it.

The activities suggested in both the Investigation and the Worksheets should be spread out over several sessions in order to allow the students sufficient time to work through each and try different ways and approaches. The best strategies evolve after initial attempts are exhausted. Notes accompanying the activities suggest ways to investigate the activities more deeply from a different perspective, or to extend the ideas developed.

Comments: Specific for the Investigation

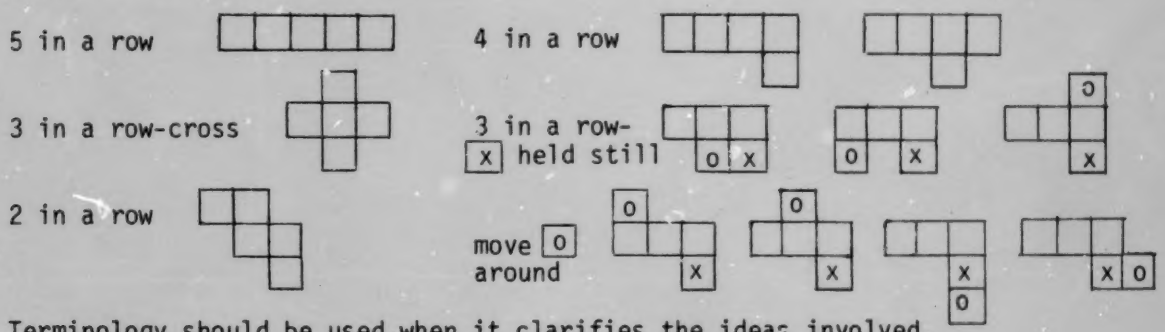
1. The 5 possible arrangements are:



Illustrate the condition that whole sides must correspond by relating it to the way postage stamps are attached. The two figures shown are mirror images of each other.

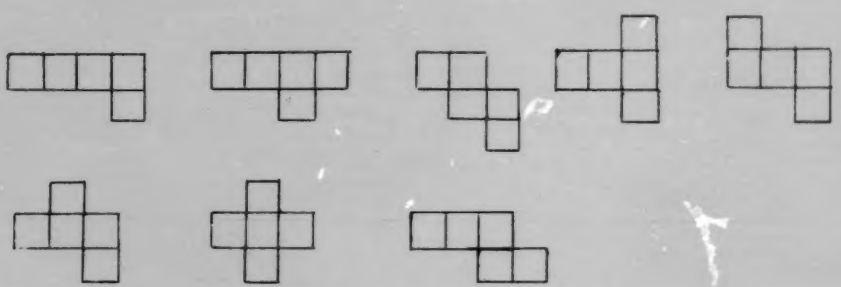
2. Use crayons or felt pens to make the outlines of the shapes easier to see on the grid paper. Younger children may need to place the squares directly on the grid paper rather than using a mental image to reproduce it.

3. At first, most children will use trial and error to find the different arrangements. This leads to making the same arrangement more than once in a different orientation. Use the cutouts to check these for congruency by using a turn and/or flip to make one fit on top of the other. When they are convinced that the trial and error method does not guarantee that all the possibilities are found, suggest they use a systematic strategy to make sure none are overlooked. One strategy is suggested here:



4. Terminology should be used when it clarifies the ideas involved.

5. This is an introduction to the process of making 3 dimensional shapes from 2 dimensional patterns or "nets" (see the "Solid Shapes Investigation"). It emphasizes that the faces of solids are plane figures. The box makers are:



PENTOMINOES

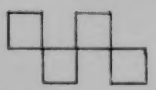
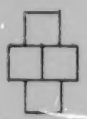
Investigation

1. Take 4 squares from the box. See how many different shapes you can make by arranging them with their sides joining. Two ways are different if one cannot be made to fit on top of the other.

   So this: is not different from this:

   Do you see why?

   Remember, the squares must have complete sides touching. These are not permitted:

2. Record the different arrangements of grid paper. Did you find five different ways?

3. Now take a 5th square. How many different ways can you arrange these 5 squares? Record the different arrangements on a piece of grid paper. Cut them out and check that no two are congruent. Two shapes are congruent if one can be made to fit exactly on top of the other.

4. These twelve different arrangements of five squares are called "pentominoes". Why do you think they are called this? What is a pentagon? How is it like the pentominoes? What are dominoes? How are they like the pentominoes?

5. Imagine a box, all of whose sides are squares. Take the top off your box. How many sides does your box have now? Each of your pentominoes has five sides, too. Which of your pentominoes do you think could be folded to make a box without a lid? Just by looking and predicting, make one pile of those you think are box-makers, and one pile of those you think are not box-makers. Now find out if you were right by folding them. How many are box-makers?

6. This is not an easy task, and should be reserved for the older children. Worksheet #4 provides a lead into this activity.

7. This is a possible extension which the younger will play by trial and error, but on which the older children should soon use strategies.

   Extension: Finding all 35 hexominoes makes an excellent growing bulletin board that students can add to as new arrangements are found. Only 11 fold to make a cube. (Interestingly, although there are over 2,000 ways to form a rectangle with a set of pentominoes, it is impossible to make a rectangle with a set of hexominoes.) For the proof of this, see Gardner, op. cit.

6. Using your whole set of 12 pieces, can you fit them together to form a rectangle? How many different ways can you find to make a rectangle? Can you make one that is 5 squares wide and 12 squares long, and another that is 6 squares wide and 10 squares long? (There are over 2,000 ways to arrange the pieces into a 6 by 10 rectangle!)

7. Get together with a friend and play "Pentominoes". You will need one 8 by 8 game board and one set of 12 pentominoes.

   Rules:

   a) The first player places a pentomino anywhere on the game board, exactly covering 5 squares.

   b) Then the second player does the same. The shapes cannot overlap.

   c) The players alternately continue to place shapes on the board until a player cannot play, either because there are no more shapes, or there is no suitably shaped space on the board.

   d) The last to be able to play is the winner.

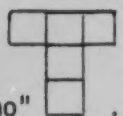
   e) Play several games. Is there always a way to win?

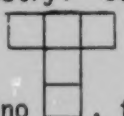
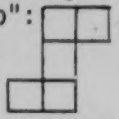
   (Solitaire Pentominoes: Can you arrange all 12 shapes on the board? If not, the board wins. If you can, you win!)

   Challenge: How many different hexominoes (using 6 squares) can you find?
   How many will fold to form a cube?

PENTOMINOES

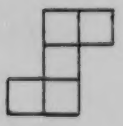
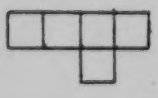
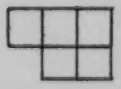
Worksheet #1

1. Take the "T-Pentomino"  . Can you fold it in half so that its parts match exactly? This works because it has a line of symmetry: the line on which it can be folded so its parts match exactly. Look carefully at each of the other pentominoes. Sort into one pile those you think have a line of symmetry, and into another pile those that do not. Now test each piece by folding. Were you right? Does any pentomino have more than one line of symmetry? See chart below.

2. Trace around the T-Pentomino  , then turn the piece on top of its tracing, leaving the tracing still. It will not match its tracing until it has been turned completely around and is back in its original position. Now try the "S-Pentomino":  .Trace it, then turn the piece on top of the tracing.

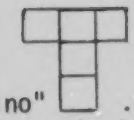
   Did you find that this piece matched its tracing when it is turned half way? This happens because the S-Pentomino has turn symmetry: it matches itself (or its tracing) when turned part of a full turn. Sort, then check the other pentominoes for turn symmetry. See chart below.

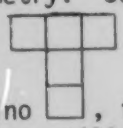
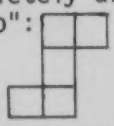
3. Record your findings in the chart:

Pentomino	Has Line Symmetry	Has Turn Symmetry
	Yes	No
	No	Yes
	Yes (Has 2 lines: one horizontal, one vertical.)	Yes
	No	No
	No	No
	No	No

PENTOMINOES

Worksheet #1

1. Take the "T-Pentomino"  . Can you fold it in half so that its parts match exactly? This works because it has a line of symmetry: the line on which it can be folded so its parts match exactly. Look carefully at each of the other pentominoes. Sort into one pile those you think have a line of symmetry, and into another pile those that do not. Now test each piece by folding. Were you right? Does any pentomino have more than one line of symmetry? See chart below.

2. Trace around the T-Pentomino  , then turn the piece on top of its tracing, leaving the tracing still. It will not match its tracing until it has been turned completely around and is back in its original position. Now try the "S-Pentomino":  .Trace it, then turn the piece on top of the tracing.

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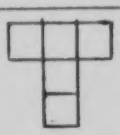
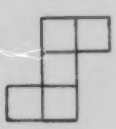
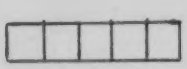
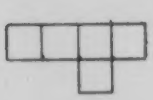
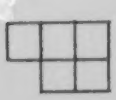
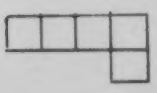
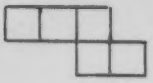
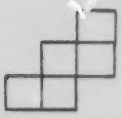
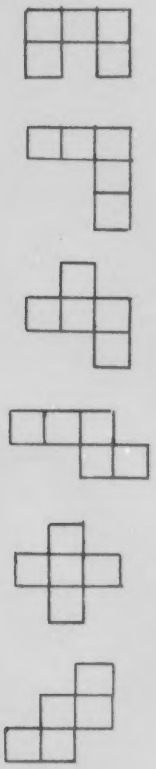
	Yes	No
	No	Yes
		
		
		
		

Pentomino	Has Line Symmetry	Has Turn Symmetry
	Yes	No
	No	No
	No	No
	No	No
	Yes (2 lines of symmetry)	Yes
	Yes	No

4. Do some of the shapes have both line and turn symmetry? What do you notice about them when this occurs?

   If a shape has two (or more) lines of symmetry, it also has turn symmetry.

Pentomino	Has Line Symmetry	Has Turn Symmetry
		

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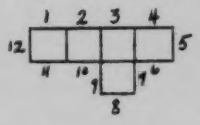
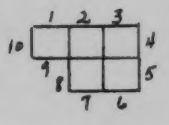
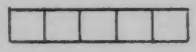
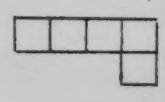
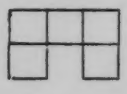
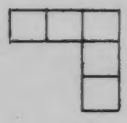
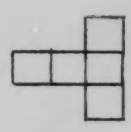
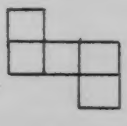
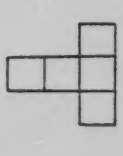
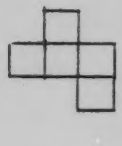
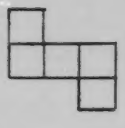
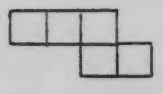
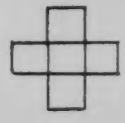
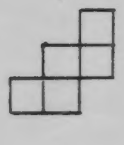
PENTOMINOES

Worksheet #2

1. Since each of the pentominoes is made from 5 squares, each has an area of 5 square units; that is, all the pentominoes have the same area. How do their perimeters compare? Remember, the perimeter of a figure is the distance around it. Counting the side of each square as 1 unit, find the perimeter of each pentomino, and fill in the chart. The first two have been done for you.

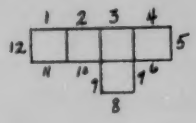
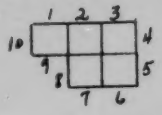
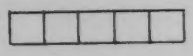
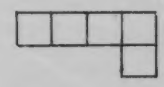
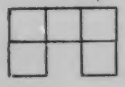
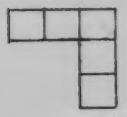
Pentomino			
Perimeter	12	10	12
Pentomino			
Perimeter	12	12	12
Pentomino			
Perimeter	12	12	12
Pentomino			
Perimeter	12	12	12
Pentomino			
Perimeter	12	12	12

2. What do you notice? Do shapes with equal areas always have equal perimeters?
   Shapes with the same area may have different perimeters.

PENTOMINOES

Worksheet #2

1. Since each of the pentominoes is made from 5 squares, each has an area of 5 square units; that is, all the pentominoes have the same area. How do their perimeters compare? Remember, the perimeter of a figure is the distance around it. Counting the side of each square as 1 unit, find the perimeter of each pentomino, and fill in the chart. The first two have been done for you.

Pentomino			
Perimeter	12	10	____
Pentomino			
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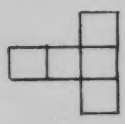
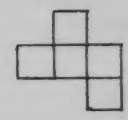
Pentomino			
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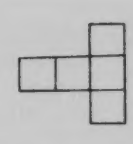
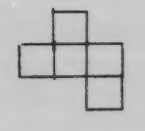
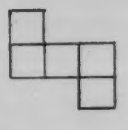
Pentomino			
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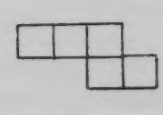
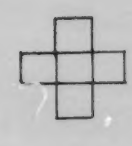
Pentomino			
Perimeter	____	____	____

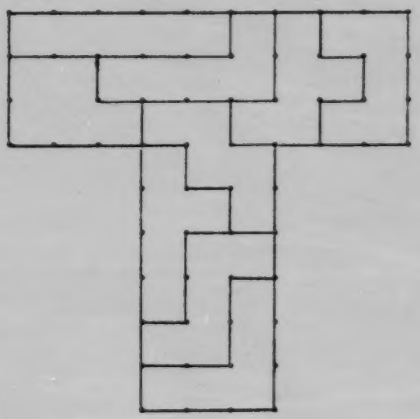
2. What do you notice? Do shapes with equal areas always have equal perimeters?

PENTOMINOES

Worksheet #3

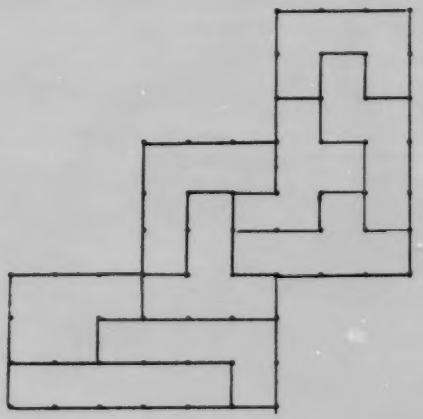
1. Take the T-Pentomino:

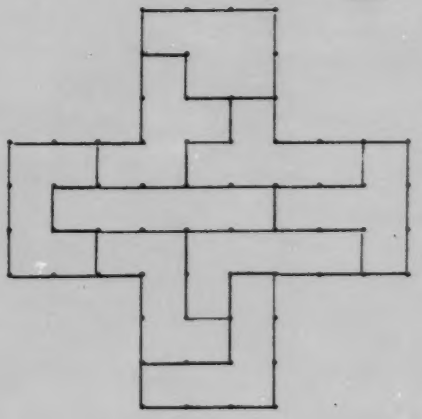
   Use nine of the remaining pentominoes to make an enlarged scale model of the T-Pentomino. (Some of the pieces have been placed for you.)



2. What is the area of the enlarged model? How does this area compare with the area of the T-Pentomino? What are the lengths of each of the sides of the enlarged model? How do these lengths compare with the corresponding sides of the T-Pentomino?

3. Repeat this "triplication" process for the and the





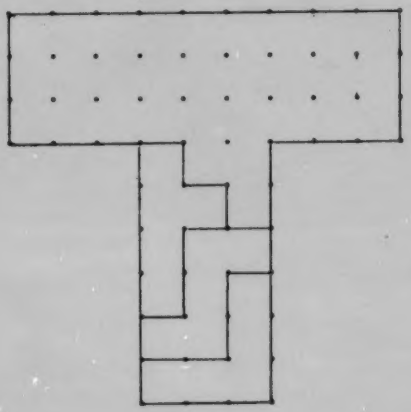
4. What other pentominoes can you triplicate?

PENTOMINOES

Worksheet #3

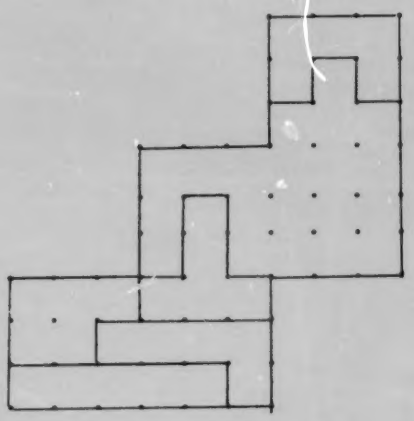
1. Take the T-Pentomino:

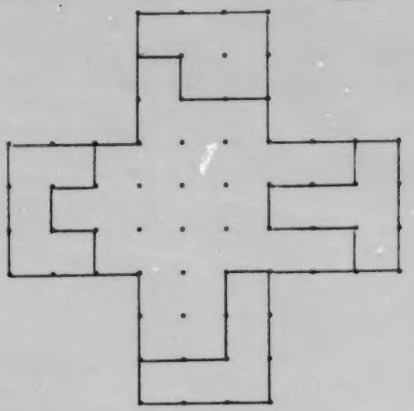
   Use nine of the remaining pentominoes to make an enlarged scale model of the T-Pentomino. (Some of the pieces have been placed for you.)



2. What is the area of the enlarged model? How does this area compare with the area of the T-Pentomino? What are the lengths of each of the sides of the enlarged model? How do these lengths compare with the corresponding sides of the T-Pentomino?

3. Repeat this "triplication" process for the and the



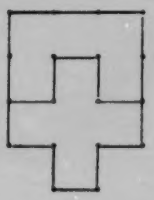


4. What other pentominoes can you triplicate?

PENTOMINOES

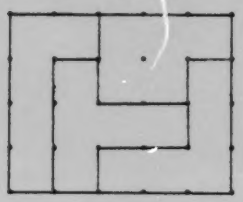
Worksheet #4

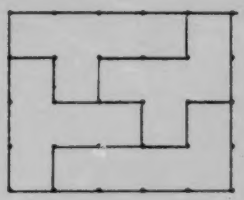
1. Fit 2 pentominoes into this shape. How many ways can you find to do this?



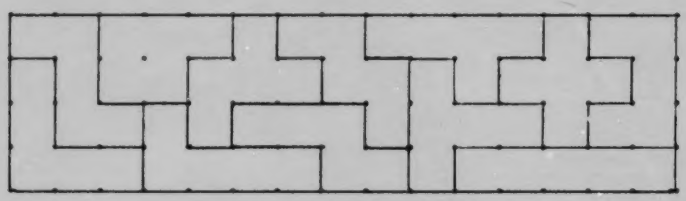
4 different solutions possible.

2. Fit any 4 pentominoes into each of these rectangles. One is placed for you.

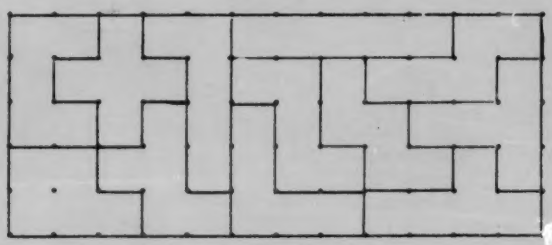




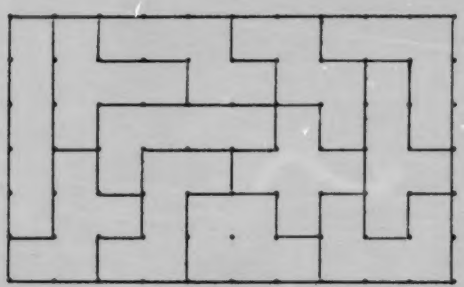
3. Each of the rectangles below has some of a set of 12 pentominoes drawn in. Complete each rectangle by placing the remaining pentominoes to form a rectangle using all 12 pentominoes.



(A 15 x 4 rectangle, 5 pieces missing.)



(A 12 x 5 rectangle, 6 pieces missing.)



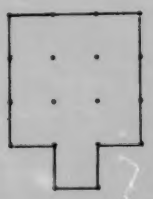
(A 10 x 6 rectangle, 6 pieces missing.)

This is only one of over 2,000 possible solutions.

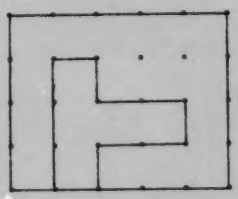
PENTOMINOES

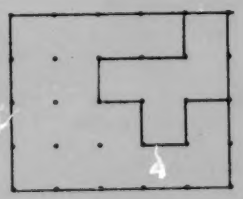
Worksheet #4

1. Fit 2 pentominoes into this shape. How many ways can you find to do this?

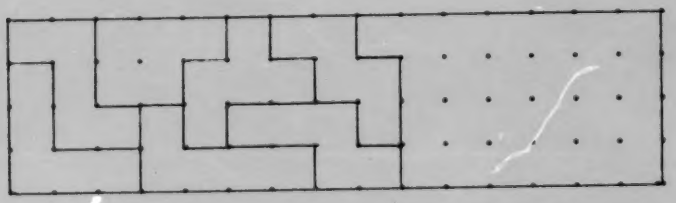


2. Fit any 4 pentominoes into each of these rectangles. One is placed for you.

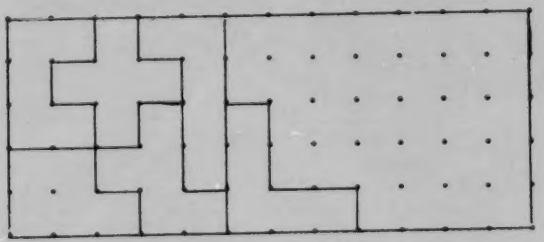




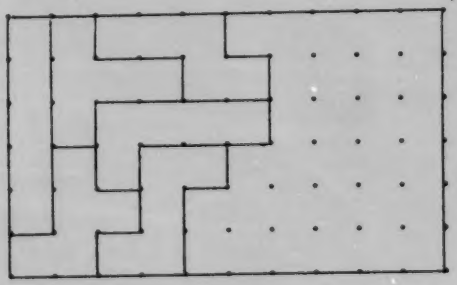
3. Each of the rectangles below has some of a set of 12 pentominoes drawn in. Complete each rectangle by placing the remaining pentominoes to form a rectangle using all 12 pentominoes.



(A 15 x 4 rectangle, 5 pieces missing.)



(A 12 x 5 rectangle, 6 pieces missing.)



(A 10 x 6 rectangle, 6 pieces missing.)

PROBABILITY

Notes to the Instructor

Objectives:
- to lay foundations for later more formal work by informally developing the concept of probability as the ratio of the number of ways success can occur to the total number of ways the event can happen;
- to develop counting techniques for determining probabilities;
- to develop skills in conducting experiments, recording data, and interpreting results;
- to experience making probabilistic predictions and correlating them with observed outcomes;
- to develop the idea of randomness of happenings;
- to experience that the outcomes of repeated trials of the same experiment are independent of previous results.

Materials:
- Coins: one per student.
- Die: one per student.
- Paper and pencils: one per student.
- Bag with 10 marbels: 3 green, 1 red, 6 blue.

References:

CEMREL. Elements of Mathematics, Book 0: Intuitive Background. St. Ann, MO: CEMREL, Inc., 1971 (Chapter 8: "An Introduction to Probability").

Fouch, Daniel J. Activities in Elementary Probability: MCTM Monograph No. 9, September 1975.

Gardner, Martin. "Probability Paradoxes." In The Scientific American Book of Mathematical Puzzles & Diversions. New York: Simon and Schuster. 1959: 47-56.

Nuffield. Probability and Statistics. New York: John Wiley and Sons, Inc., 1969.

Page, David A. "Probability" in The Growth of Mathematical Ideas, Grades K-12. 24th Yearbook of National Council of Teachers of Mathematics. Washington, D.C.: The Council. 1969.

"Probability": in More Topics in Mathematics for Elementary School Teachers. 30th Yearbook of National Council of Teachers of Mathematics. Washington, D.C.: The Council. 1969.

Shulte, Albert P. and Stuart A. Choate. What Are My Chances? Book A and Book B, Palo Alto, CA: Creative Publications. 1977.

Yee, Albert H. "Mathematics, Probability and Decision Making." The Arithmetic Teacher. (May 1966): 385-387.

Comments: General

Applications of probability are numerous in the students' lives. Many daily activities, such as playing games or dressing according to the weather forecast involve probability since they take place in the face of uncertainty. Probability theory is a mathematical way to study the uncertainties of daily affairs.

Students are naturally intrigued by ideas of uncertainty and chance. This Investigation focuses their thinking about probability requiring them first to think about an experiment and predict its result, then to perform it, and finally to compare the results with their predictions. In this way they are experiencing in an intuitive way a fundamental idea of theoretical probability: namely, that probability is what is expected to happen in the long run, rather than a certain answer to a specific situation. That is, probability provides a way of theoretically predicting what can be expected; actual occurrences only approximate the theoretical expectation.

The experiments in the Investigation ask the students to predict the results of an experiment based on theoretical probabilities. In other cases no theoretical model is available, and an experimental probability must be used, based on the actual outcomes of the experiement. An example of an activity to illustrate the process of determining the probability, given the data, is the following:

> Use a coffee can with a lid, and with 3 holes in the bottom cut just large enough so that the color of the marbel directly over the hole may be observed without the marble falling through the hole. In the can, place 12 marbles: 8 white, 4 black. Tell the students there are 12 marbles, some black, some white. Have them shake the can, record the number of black marbles showing, and repeat this 10 then 20 times. From the results have them predict the number of black marbles in the can, reminding them there are only 12 marbles in the can. Extend this by using the experimental probability (the ratio of the number of black marbles seen to the total number of marbles seen) to predict the outcome of a given number of shakes of the can and checking how good the prediction is.

You might want to bring in a discussion of sampling techniques at this time. A good shake of the can should give a random sample. A large random sample is usually a good representation of the whole; however, it must be regarded as a possible though improbable occurrence that such a sample is misleading.

This Investigation is only a brief introduction to some of the fundamental, intuitive ideas of probability that are suitable for developing with children. Doing experiments such as these should precede more thorough and rigorous study. Suggestions for such further work are included in the accompanying Specific Notes.

Comments: Specific for the Investigation

1. Since half of the ways a coin can land show heads, it is expected that half of the tosses will show heads. Students will find that the actual or experimental result of exactly half the tosses showing heads is rare, but that as the number of tosses increases, the actual results come closer to the theoretical expectation.

2. The theoretical probability of heads = $\frac{\text{number of ways heads can show}}{\text{total number of ways a coin can land}} = \frac{1}{2}$

3. a. The theoretical probability of a 3 is $\frac{1}{6}$ (written $P(3) = \frac{1}{6}$). Since $\frac{1}{6}$ of 12 = 2, it is expected a 3 would show twice.

   b. It is just as likely a 3 will show as a 4; these are equally likely events. The fact that 4 is larger numerically does not affect its probability.

   c. Introduce the standard method of tallying (recording five as for easier counting) or have students invent other methods of keeping track. Develop the idea of randomness of events by noting the random way tallies appear in the columns. The students should gain from this a feeling for equally likely events since the "chances" of one number showing on a single roll are the same as any other number; that is, the six possible outcomes of a single roll are equally likely.

PROBABILITY

Investigation

1. For a coin, if you toss it, it could land either heads or tails. If you toss the coin 6 times, how many heads do you expect to get? Do you expect half of the 6 tosses to land heads? Try it. Did you get 3 heads? Try it again. What did you get this time? Now toss the coin 50 times, keeping a record of the sums of the results as you work on a table like this:

Total Number of Tosses	Number of Heads Expected	Actual Number of Heads
10		
20		
30		
40		
50		

What did you notice? When did what you actually got come closest to what you expected?

2. You were using probability to figure the expected number. Why do you think we say the probability of getting heads when a coin is tossed is 1 out of 2, or ½?

3. Now take a die from the packet.
   a. How many faces has it? What numbers show on each face? If you roll the die, it could land showing a 1, a 2, a 3, a 4, a 5, or a 6. What do you think is the probability of showing a 3 when the die is rolled? If you roll it 12 times, how many times would you expect the 3 to show? Why? Now try it. Did what you expected happen?
   b. If you roll the die 12 times, how many times would you expect the 4 to show? Are the chances of showing a 4 the same as showing a 3? Why or why not?
   c. Make a chart like the one below. Fill in the second column with the number of times you would expect each number to show if the die is rolled 60 times. Now roll the die 60 times keeping record of the results as you go using tallies.

Number on Top Face	Expected Number of Times Showing	Recording Tallies	Actual Number of Times Showing
1			
2			
3			
4			
5			
6			

Did you actually get the results you were expecting? Were your results close?

d. This alludes to the frequently encountered but fallacious "Law of Averages". The result of a roll is not affected by the results of previous rolls. There is still a $\frac{1}{6}$ probability of a 3 on the 20th roll even if the first 19 rolls did not show a 3, just as there is still only a ½ probability (approximately) of the next child being a girl in a family of a large number of consecutive boys.

e. P (even) = $\frac{3}{6}$ or $\frac{1}{2}$ since there are 3 even numbers out of 6 numbers altogether.

f. It is helpful to actually list the ways in order to count them. P (number less than 5) = $\frac{4}{6}$ or $\frac{2}{3}$ since there are four ways a number less than five can show.

g. P (number less than 1) = $\frac{0}{6}$ or 0.

P (Number 1 to 6) = $\frac{6}{6}$ or 1.

Thus the probability of an impossible event is 0 and that of a certainty 1. All other probabilities are between 0 and 1.

4. P (green) = $\frac{3}{10}$

P (red) = $\frac{1}{10}$

P (blue) = $\frac{6}{10}$ or $\frac{3}{5}$

Since $\frac{1}{10} < \frac{3}{10} < \frac{6}{10}$, the most likely event is drawing a blue. Since the marble is replaced after each draw, these probabilities hold for each of the 20 trials.

d. If we made only 6 rolls, would it have to be that each number shows once? After 7 rolls, could one of the numbers still have not shown? After 10 rolls? After 20 rolls? If after 20 rolls, one of the numbers still has not shown, is it pretty certain that it will show on the next throw?

e. Look at your chart. What is the total number to times an even number showed? What part of the total number of tosses (60) is this? Did an even number show about half the time? What is the probability of an even number showing when a die is rolled?

f. How many ways can a number less than 5 show if a die is rolled? What are they? Use this to tell the probability of getting a number less than 5.

g. What is the probability of getting a number less than 1 when a die is rolled? (Hint: how many faces shows a number less than 1?) If something can't happen, what is its probability? What is the probability of getting a 1, a 2, a 3, a 4, a 5, or a 6 when a die is rolled? If something is certain to happen, what is its probability?

4. Now take the bag with 10 marbles. There are 3 green, 1 red and 6 blue marbles in the bag. If you close your eyes and draw a marble from the bag, what color will it most likely be? Why? What is the probability of drawing a green marble? The probability of a red marble? The probability of a white marble? Draw a marble, note its color, return it to the bag, shake the bag, and draw again. Repeat this 20 times, and record your results in the chart.

Color	Expected Number	Actual Number
Green		
Red		
Blue		

How do you explain your results?

PROBABILITY

Worksheet #1

1. You found that the probability of getting a 4 when a die is rolled is $\frac{1}{6}$ since 1 out of 6 of the faces of the die shows a 4. This can be written P (4 on a die) = $\frac{1}{6}$ or, if you know a die is being used, P (4) = $\frac{1}{6}$.

   What do these short forms mean?

   P (5 on a die) = $\frac{1}{6}$ The probability of a 5 showing is $\frac{1}{6}$

   P (even on a die) = $\frac{3}{6}$ or $\frac{1}{2}$ The probability of getting an even number is ½.

   P (H on coin toss) = $\frac{1}{2}$ The probability of getting a head is ½.

2. A bag contains 24 marbles: 2 white, 4 red, 6 blue, 12 green. Find these probabilities for drawing one marble:

   P(white) = 2/24 = 1/12

   P(red) = 4/24 = 1/6

   P(blue) = 6/24 = 1/4

   P(not blue) = $\frac{2 + 4 + 12}{24}$ = 18/24 = 3/4 or 1-P(blue) = 1-¼ = 3/4

   P(green) = 12/24 = ½

   P(not green) = 1 - ½ = ½

   P(white or green) = $\frac{2 + 12}{24}$ = 14/24 = 7/12

   P(red or blue) = $\frac{4 + 6}{24}$ = 10/24 = 5/12

   (How does this compare with P(red) and P(blue)?) P(red-blue) = P(red) + P(blue)

   Find two ways: P(red or white or blue) = 4/24 + 2/24 + 6/24 = 12/24 = ½ or P(red or white or blue) = P(green) = ½

   HINT: How many marbles are red or white or blue?
   What do you notice about P(green) and P(not green)?

3. Suppose you toss a coin and roll a die. List all 12 possible outcomes. Find each probability.

   P(H,4) = 1/12 (Remember: this means getting heads and a 4.)

   P(T,2) = 1/12

   P(H, even number) = 3/12 = 1/4

   P(T, even number) = 3/12 = 1/4 (Can you explain this?)

   P(even number) = 6/12 = ½

   P(H) = ½

   P(3) = 1/6

   P(H,3) = ½ (What do you notice about the last three probabilities?)

   P(H,3) = P(4) x P(3)

PROBABILITY

Worksheet #1

1. You found that the probability of getting a 4 when a die is rolled is $\frac{1}{6}$ since 1 out of 6 of the faces of the die shows a 4. This can be written P (4 on a die) = $\frac{1}{6}$ or, if you know a die is being used, P (4) = $\frac{1}{6}$.

   What do these short forms mean?

   P (5 on a die) = $\frac{1}{6}$

   P (even on a die) = $\frac{3}{6}$ or $\frac{1}{2}$

   P (H on coin toss) = $\frac{1}{2}$

2. A bag contains 24 marbles: 2 white, 4 red, 6 blue, 12 green. Find these probabilities for drawing one marble:

   P(white) =

   P(red) =

   P(blue) =

   P(not blue) =

   P(green) =

   P(not green) =

   P(white or green) =

   P(red or blue) =
   (How does this compare with P(red) and P(blue)?)

   Find two ways: P(red or white or blue) =

   HINT: How many marbles are red or white or blue?
   What do you notice about P(green) and P(not green)?

3. Suppose you toss a coin and roll a die. List all 12 possible outcomes. Find each probability.

   P(H,4) = (Remember: this means getting heads and a 4.)

   P(T,2) =

   P(H, even number) =

   P(T, even number) = (Can you explain this?)

   P(even number) =

   P(H) =

   P(3) =

   P(H,3) = (What do you notice about the last three probabilities?)

PROBABILITY

Worksheet #2

A red die and a green die are tossed. How many different ways could they land? List the ways by completing this table which has been started for you.

Number of Green Die

Number on Red Die	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1.5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3.1)	(3.2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Use the table to answer these questions:

1. Altogether, how many different outcomes are possible? (That is, how many ordered pairs are in the chart?) 36

   How many of these are red die 2, green die 3? 1

   What is the probability of red die 2, green die 3? P(2,3) = 1/36

2. What is the probability of getting a 2 and a 3 if it does not matter which die shows which color? 2/36 = 1/18

   Is this more likely to happen than to get them in the order red die 2, green die 3? Why? More likely since there are two ways it can happen.

3. How many ways can the dice land showing two even numbers? 9

   Find P(both even) = 9/36 = 1/4
   P(both odd) = 9/36 = 1/4

4. How many ways can the dice land showing numbers whose sum is 6? 5

   P(sum 6) = 5/36
   P(sum 12) = 1/36
   P(sum even) = ½
   P(sum greater than 4) = 30/36 = 5/6
   P(sum 5 or 7) = 4/36 + 6/36 = 10/36
   P(sum 1) = 0/36
   P(sum greater than 12) = 0

   (What do you call events like this?)
   Impossible events.

5. You are going to play a game with a friend. You take turns tossing a pair of dice. The first player wins if the sum of the numbers showing is 6 or less; the second player wins if the sum showing is 7 or more. Would you be more likely to win if you were the first player or the second player? How do you know?

   15 ways for sum to be six or less.
   21 ways for sum to be seven or more.
   Second player because the probability is 21/36 = 7/12.

PROBABILITY

Worksheet #2

A red die and a green die are tossed. How many different ways could they land? List the ways by completing this table which has been started for you.

Number of Green Die

Number on Red Die	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)			
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3						
4						
5						
6						

Use the table to answer these questions:

1. Altogether, how many different outcomes are possible? (That is, how many ordered pairs are in the chart?) _____

   How many of these are red die 2, green die 3? _____

   What is the probability of red die 2, green die 3? P(2,3) = _____

2. What is the probability of getting a 2 and a 3 if it does not matter which die shows which color? _____

   Is this more likely to happen than to get them in the order red die 2, green die 3? Why? _____

3. How many ways can the dice land showing two even numbers? _____

   Find P(both even) =
   P(both odd) =

4. How many ways can the dice land showing numbers whose sum is 6? _____

   P(sum 6) =
   P(sum 12) =
   P(sum even) =
   P(sum greater than 4) =
   P(sum 5 or 7) =
   P(sum 1) =
   P(sum greater than 12) =

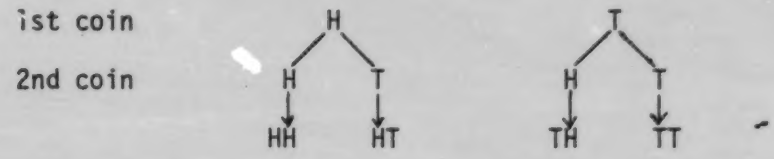
   (What do you call events like this?)

5. You are going to play a game with a friend. You take turns tossing a pair of dice. The first player wins if the sum of the numbers showing is 6 or less; the second player wins if the sum showing is 7 or more. Would you be more likely to win if you were the first player or the second player? How do you know?

PROBABILITY

Worksheet #3

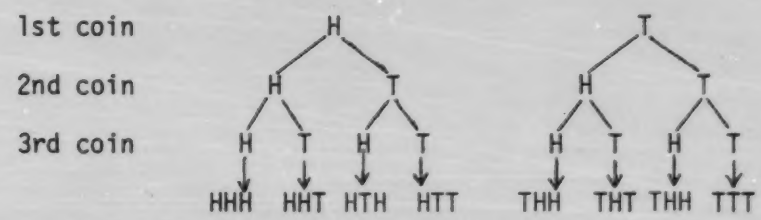
1. Two coins are tossed at the same time. What is the probability that both will land heads up? To answer this, first list all the different ways the coins can land. Use this tree diagram to list them:



So there are four ways the coins can land. How many of these ways show two heads? What is the probability of getting two heads? How many ways show a head and a tail? What is the probability of getting a head and a tail?

2/4 = ½

2. Suppose three coins are tossed. What is the probability of getting three heads? First find all the ways the coins could land by completing this tree diagram:

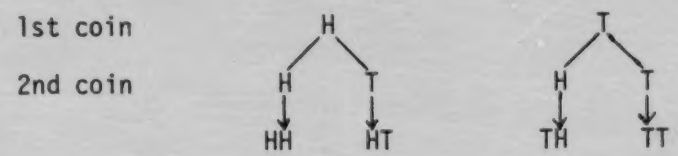


Of the 8 ways the coins could land, how many show all 3 heads? 1 So the probability of getting 3 heads is 1 out of 8, or $\frac{1}{8}$. How many ways could they show 2 heads, 1 tail? 3 (Notice that which coin shows which face does not matter.) What is the probability of getting 2 heads, 1 tail? $\frac{3}{8}$ How do you know? There are 3 ways to get 2 heads, 1 tail out of 8 ways.

PROBABILITY

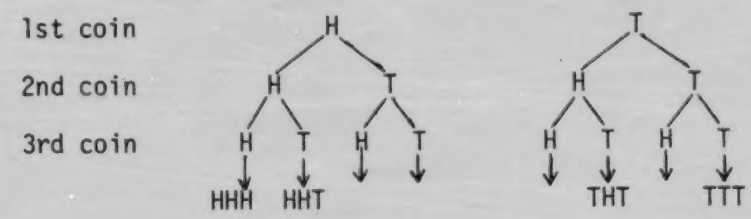
Worksheet #3

1. Two coins are tossed at the same time. What is the probability that both will land heads up? To answer this, first list all the different ways the coins can land. Use this tree diagram to list them:



So there are four ways the coins can land. How many of these ways show two heads? What is the probability of getting two heads? How many ways show a head and a tail? What is the probability of getting a head and a tail?

2. Suppose three coins are tossed. What is the probability of getting three heads? First find all the ways the coins could land by completing this tree diagram:



Of the 8 ways the coins could land, how many show all 3 heads? So the probability of getting 3 heads is _____ out of _____, or $\frac{1}{8}$. How many ways could they show 2 heads, 1 tail? (Notice that which coin shows which face does not matter.) What is the probability of getting 2 heads, 1 tail? How do you know?

SOLID SHAPES

Notes to the Instructor

Objectives: to investigate how solid shapes are built from plane shapes;
to use "Pattern finding" to find the Euler Formula that relates the number of vertices, edges, and surfaces of polyhedrons;
to examine coloring problems on two- and three-dimensional surfaces;
to provide practice in spatial perception, particularly visualizing how two-dimensional patterns form three-dimensional solids, and how three-dimensional solids can be represented in two-dimensional drawings.

Materials: Cubes: assorted sizes, 1 per student.
Squares: cut from sturdy construction paper; 3" x 3"; 6 per student.
Equilateral triangles: cut from sturdy construction paper; 4 per student.
Patterns for regular octahedron, regular dodecahedron, and regular icosahedron: duplicated on sturdy paper; 1 of each per student.
Cellophane tape.
Scissors.
Crayons.
Worksheets #1 and #2: one per student.
Grid paper: 1" x 1" grids, several sheets per student.
(Optional: copy of Coloring Game - one per pair of students.)

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Laycoch, Mary. Straw Polyhedra. Palo Alto, California: Creative Publications, 1970.
Mann, J. E. "Discovering Special Polyhedra." Mathematics Teaching, Winter, 1969, 49: 48-49.
Nuffield Mathematics Project: Shape and Size 3. New York: John Wiley & Sons, 1968: 55-59.
Shengle, C. E. "A Look at Regular and Semi-Regular Polyhedra." Mathematics Teacher, December, 1972, 65: 713-718.
Sobel, Max, and Maletsky, Evan. Teaching Mathematics: A Sourcebook of Aids, Activities, and Strategies. Englewood Cliffs, New Jersey: Prentice-Hall, 1975: 107-108.
Wardcop, R. F. "A Look at Nets of Cubes." Arithmetic Teacher, February, 1970, 17: 127-128.
Wenninger, Magnus J. Polyhedra Models in the Classroom. Reston, Virginia: National Council of Teachers of Mathematics, 1966.

Comments: General

This is an exploratory, "hands on" investigation. Children at various ages need such experiences in order to develop concepts of solid geometry. Terminology should not be overemphasized. However, the instructor should use correct terms whenever they serve to clarify an idea. Whenever possible, suitable terms which are natural to the students should be used. For example, the term "corner" is more descriptive to the students than the term "vertex" and is suitable. However, to use the term "point" is incorrect in this context,

and should be discouraged by noting that a point is a location, illustrated by a dot, and is not the same thing as a corner on a solid shape; a corner is a point, but not every point is a corner, so the terms are not synonymous.

Although in some tests, a solid is defined to be a surface with everything inside it (such as a cardboard cube filled with sand) in this investigation, the term is used to mean just the surface (the cardboard cube without the sand). The solids which are constructed are polyhedrons, since they are bounded by plane polygons (faces). The intersections of the faces are edges, and the intersections of the edges are vertices or corners. Solids with curved surfaces, such as the cylinder or sphere, are not polyhedrons.

The polyhedrons used in the investigation are regular polyhedrons: their faces are congruent regular polygons. There are only five regular polyhedrons (known as the Platonic Solids): the tetrahedron (4 faces), hexahedron or cube (6 faces), octahedron (8 faces), dodecahedron (12 faces) and the icosahedron (20 faces). A challenge to the faster students is to show why there are only five regular polyhedrons. There are many real world instances of these solids which the students will enjoy finding, such as a 12 month calendar on the faces of a dodecahedron.

For middle school students, this investigation can be extended to building semiregular solid shapes: those made from more than one type of regular polygon, such as the shape made from 6 octagons and 9 quilateral triangles. There are 14 semiregular solid shapes.

Another extension involves investigating the possible intersection of solids with a plan by "slicing" models made from clay with a piece of wire and noting the cross section.

The coloring problem at the end of the Investigation and on Worksheet #2 provides a link between solid geometry and topology, showing again the interrelatedness of the divergent areas of mathematics.

Comments: Specific for the Investigation

1. Emphasize that since a cube has 6 faces which are squares, a two-dimensional pattern (also called a net) for a cube must be an arrangment of 6 squares.* Deciding whether or not a given pattern will form a cube involves spatial perception, and students should be sure to check their conjectures by actually folding patterns. Much of mathematics involves this "predict then check" technique.

2. The 4 equilateral triangles make the tetrahedron or triangular pyramid. Some students may want to investigate possible patterns for this solid shape. The square pyramid is made with the 4 triangles and a square base (assuming the lengths of the sides of the square and the triangles are equal).

3. Intermediate grade students enjoy building the solid shapes themselves, thus learning how certain plane shapes fit together to form solid shapes. Some will be stimulated to look at the shapes and construct patterns as done above with the cube. For younger students, have the solid shapes already constructed. For the constructions, be sure the patterns are on sturdy paper. The patterns included here have construction "tabs" omitted in order to emphasize the shape of the polygons which become the faces of the solids.

4. It helps younger children to develop the abstract concepts of point line and plane by actually feeling the corners, edges, and faces of polyhedra models.

5. The table will yield Euler's Formula: Number of Faces + Number of Corners = Number of Edges + 2. The emphasis is on "pattern finding", not memorizing the formula.

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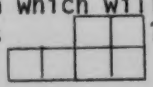
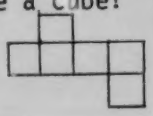
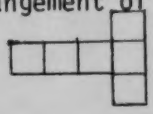
*See The Pentominoes Investigation for further activities involving arranging squares in different ways.

SOLID SHAPES

Investigation

1. Take a cube from the packet. Look at it carefully from all directions. The flat sides are called faces. How many faces are there? What shape is each? Take 6 squares from the packet. Tape them together to make a cube. Now carefully unfold your cube by cutting just enough of the taped edges so the six squares lie flat. This gives you a pattern which folds to make a cube. What does your pattern look like? Does your friend's pattern look like yours? Do you think any arrangement of 6 squares is a pattern which will fold to make a cube? Is  a pattern for a cube? Is  ? What about  ?

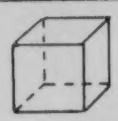
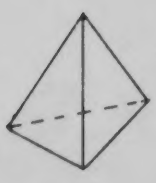
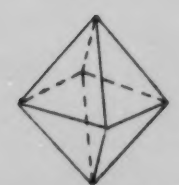
   Draw as many different patterns for a cube as you can. Test any you are not sure about by drawing the pattern on grid paper, cutting the pattern out, and trying to fold it into a cube. How many patterns for cubes did you find? Can you tell a six-square arrangement that is a cube-maker from one that isn't without folding? How?

2. Take four equilateral triangles from the packet. Can you tape them together to make a solid shape? Could a solid shape be made with four triangles and a square? Try it and see.

3. Take the patterns for three other solid shapes from the packet. Cut them out carefully along the solid lines. Then carefully fold them along the dotted lines, and tape the edges to form solid shapes.

4. Now investigate your solid shapes. What are the flat surfaces (<u>faces</u>) of each? Which have faces that are triangles? Can you name the shape of the faces of each? Pick one of your solid shapes, and run your finger along the part where two faces meet. This is called an <u>edge</u>. As your finger goes from one edge to another, it goes around a <u>corner</u>.

5. What do you notice about these solid shapes? Count the number of faces, the number of edges, and the number of corners of each, and see what you discover. Make a table like this, and fill in the numbers:

Shape	Number of Faces	Number of Corners	Number of Edges
			
			
			

6. Euler's Formula* will hold for all polyhedrons, but not for solids with curved surfaces. Have the children suggest why this is so. Be sure to include non-regular polyhedrons in this investigation.

7. This is a lead-in to the Four Color Map Problem, in Worksheet #2. At this point, it will be attempted on a trial and error basis. Emphasize that the problem is to find the fewest number of colors needed.

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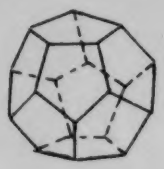
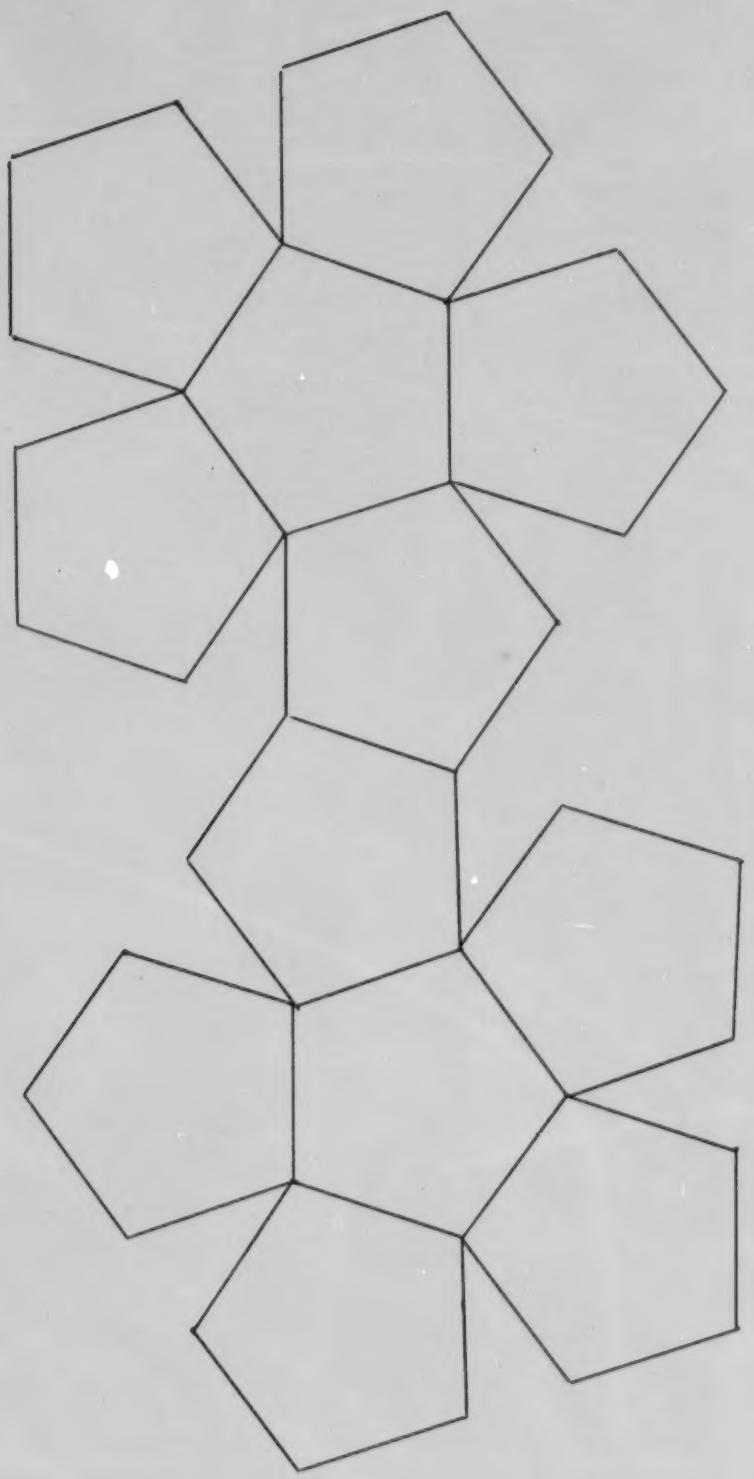
*See the Topology Investigation for the application of this same formula to networks.

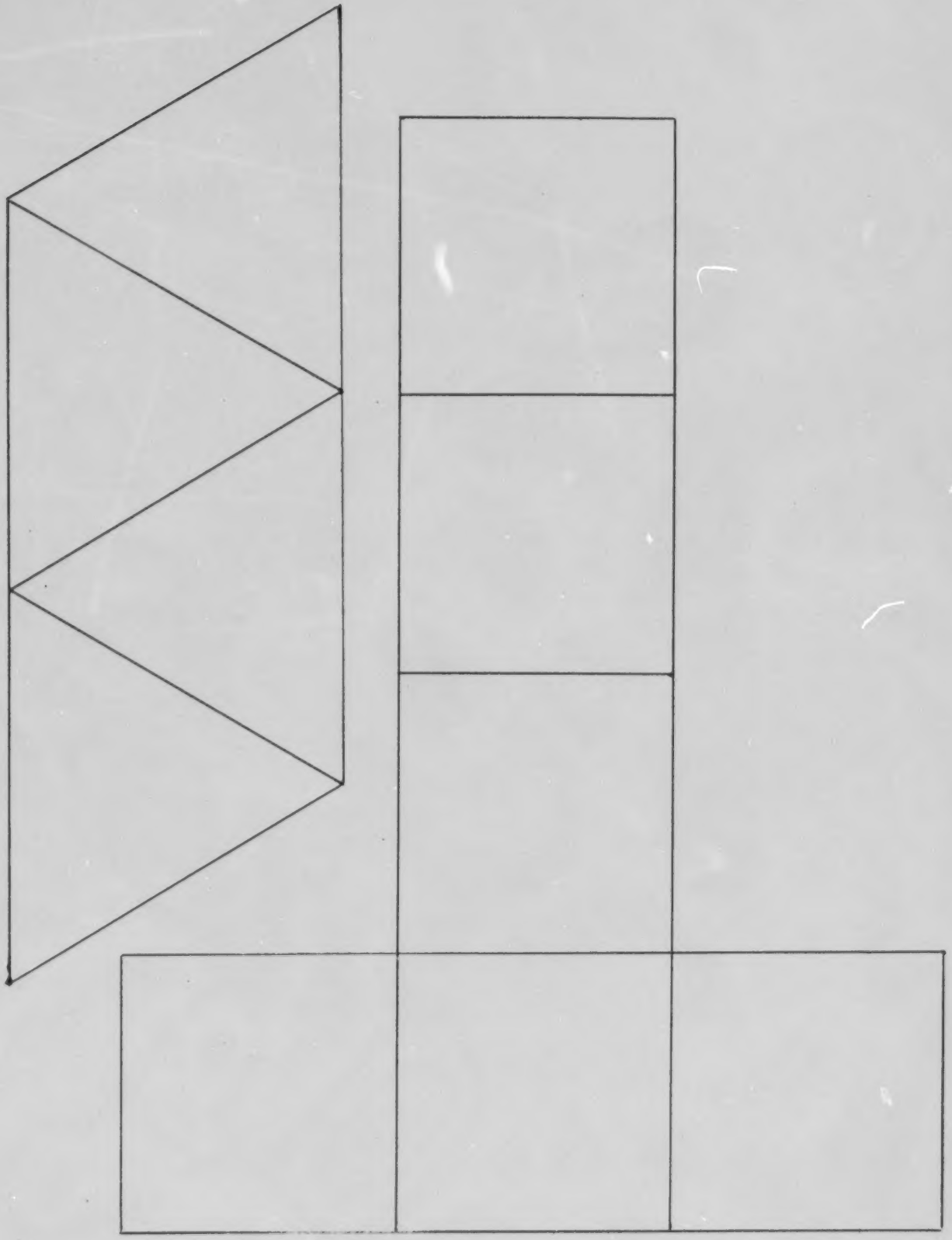
5.

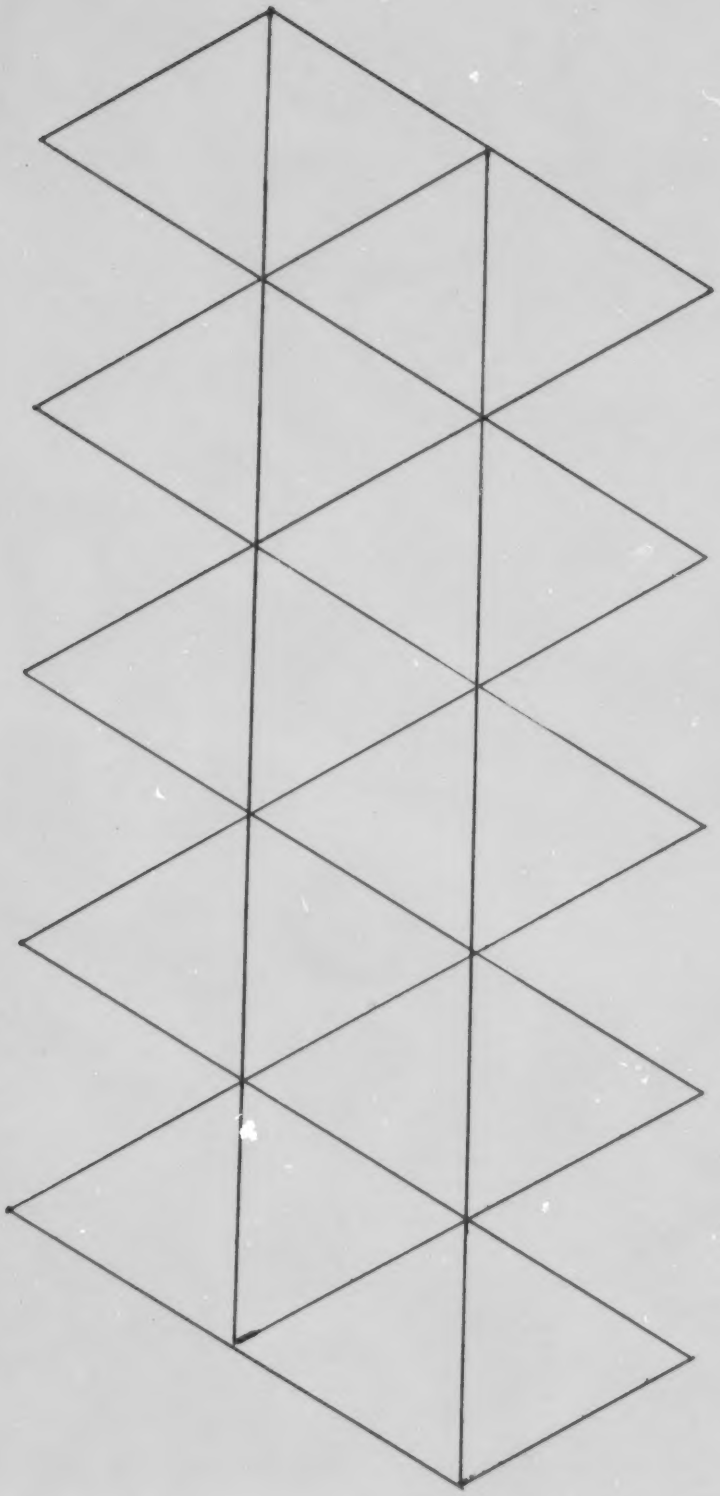
Shape	Number of Faces	Number of Corners	Number of Edges
			

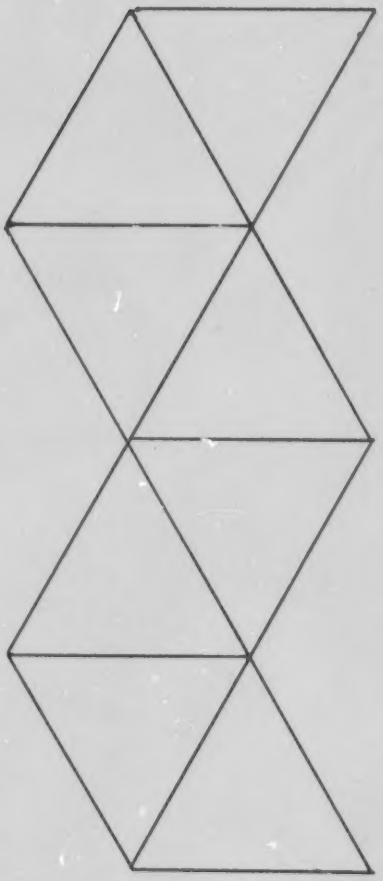
6. Look around and find other solid shapes. Does the pattern you found hold for them too? Test it with solid shapes you find at home.

7. With the shapes you made, try to color the faces so that no two faces which share an edge are the same color. What is the fewest number of colors you need to do this? Does it depend on the shape?







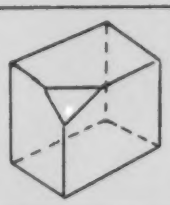


SOLID SHAPES

Worksheet #1

1. You found a relationship among the number of faces F, the number of corners C, and the number of edges E of some solid shapes. The relationship expressed in symbols is F + C = E + 2. How would you say this in words? It was found by a Swiss mathematician, Leonard Euler, in the 18th century, and is called Euler's Formula.

2. Do you think Euler's Formula will hold for all shapes whose faces are flat? Try to visualize these shapes, and fill in the chart. In the drawings, what does a dotted line mean?

Shape	Number of Faces, F	Number of Corners, C	Number of Edges, E	Euler's Formula holds? (yes,no)
	7	10	15	7 + 10 = 15 + 2 Yes
	5	5	8	5 + 5 = 8 + 2 Yes
	8	12	18	8 + 12 = 18 + 2 Yes
	7	10	15	7 + 10 = 15 + 2 Yes

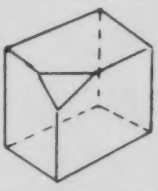
3. Do you think Euler's Formula will work for solid shapes that do not have flat faces? Find two different ones with curved faces and investigate them. What do you observe? Does not hold.

SOLID SHAPES

Worksheet #1

1. You found a relationship among the number of faces F, the number of corners C, and the number of edges E of some solid shapes. The relationship expressed in symbols is F + C = E + 2. How would you say this in words? It was found by a Swiss mathematician, Leonard Euler, in the 18th century, and is called Euler's Formula.

2. Do you think Euler's Formula will hold for all shapes whose faces are flat? Try to visualize these shapes, and fill in the chart. In the drawings, what does a dotted line mean?

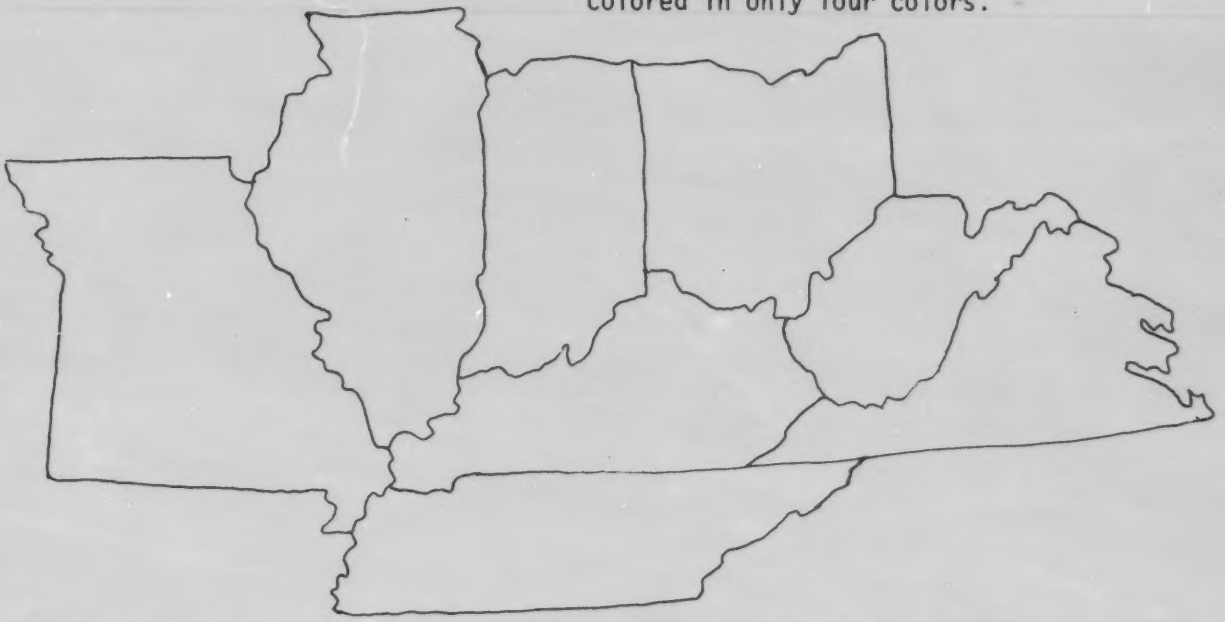
Shape	Number of Faces, F	Number of Corners, C	Number of Edges, E	Euler's Formula holds? (yes,no)
				

3. Do you think Euler's Formula will work for solid shapes that do not have flat faces? Find two different ones with curved faces and investigate them. What do you observe?

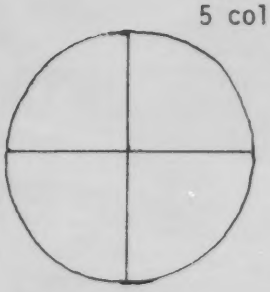
SOLID SHAPES

Worksheet #2

1. When coloring your solid shapes with the fewest colors such that no two faces next to each other had the same color, you were working on a problem in three dimensions that map makers have been working on in two dimensions for centuries. Countries or states which have a border in common must be different colors; if they have only a point in common they may be the same color. Color this map using the fewest colors possible. How many did you need? Four colors are needed. Any map can be colored in only four colors.

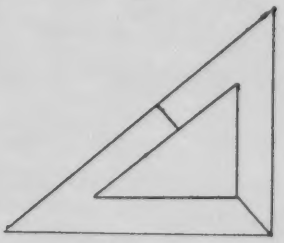


2. Pretend that these are maps. Using the same rules, that no two countries can be the same color, what is the fewest colors you need for each? Some take only 2 colors, some 3, some 4. Can you make a map that takes 5 colors? No

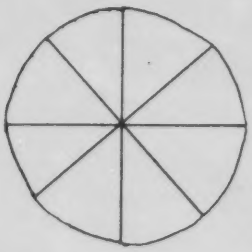


2 colors

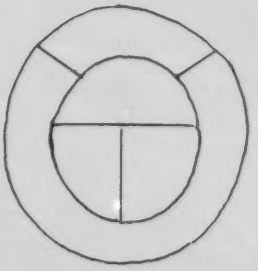
4 colors



3 colors



2 colors



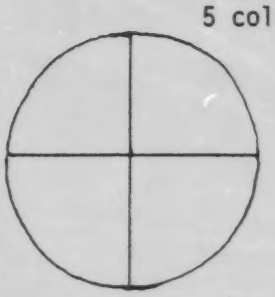
4 colors

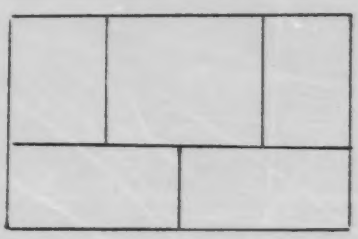
SOLID SHAPES

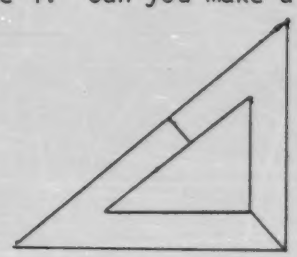
Worksheet #2

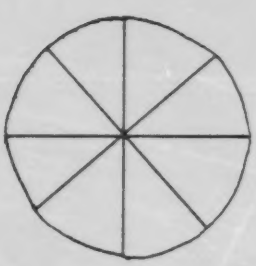
1. When coloring your solid shapes with the fewest colors such that no two faces next to each other had the same color, you were working on a problem in three dimensions that map makers have been working on in two dimensions for centuries. Countries or states which have a border in common must be different colors; if they have only a point in common they may be the same color. Color this map using the fewest colors possible. How many did you need?

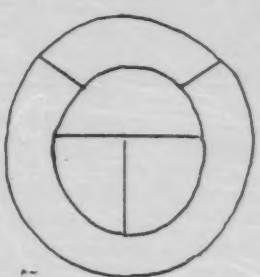
2. Pretend that these are maps. Using the same rules, that no two countries can be the same color, what is the fewest colors you need for each? Some take only 2 colors, some 3, some 4. Can you make a map that takes 5 colors?











The Map Coloring Game

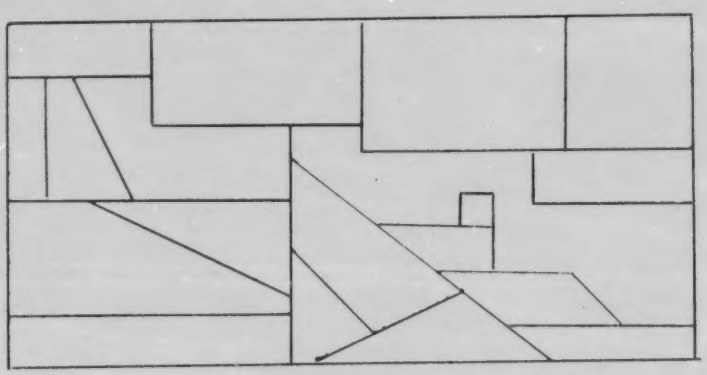
This game is played by two people. The object is to color the shapes so that your opponent cannot finish coloring the map using only 4 colors.

First player: Fill in as many shapes as you like using 4 or fewer colors. The shapes you choose cannot be bordering each other.

Second player: Try to finish coloring the shapes so that all the shapes are colored but no two shapes next to each other are the same color. Use only the same four colors as the first player.

If the first player colors the shapes so that the second player must use a fifth color to finish the map, the first player wins.

If the second player can finish coloring the map with only the same four colors, the second player wins.



(Another way to play: Take turns coloring one shape at a time, using any one of four colors. Try to force your opponent to have to use a fifth color in order to color a shape so that it is not the same color as a shape it is next to.)

SYMMETRY

Notes to the Instructor

Objectives: to investigate line symmetry of familiar figures using the fold test and the mirror test;
to investigate turn symmetry of figures using the "trace and match the tracing" test;
to become aware of symmetry as a balance of parts in objects of the environment.

Materials: Construction paper cutouts:
one per student - rectangles (3" x 5")
equilateral triangles (3")
squares (4" x 4")
parallelograms (2" x 6")
Paper: unlined; three sheets per student.
Scissors: approximately one pair for each two or three students.
Mirrors: small field mirrors; one per student.
Worksheet #1: one per student.
Worksheet #2: one per student.

References: Minnemast (Minnesota School Mathematics and Science Teaching Project), Symmetry Units 7 and 14. Minneapolis: University of Minnesota, 1968.
Nuffield Mathematics Project. Shape and Size 3. New York: John Wiley and Sons, Inc., 1968, pp. 23-25.
Walter, Marion. "An Example of Informal Geometry: Mirror Cards." Arithmetic Teacher. (October, 1966), pp. 448-462.

Comments: General

Even young children are already familiar with the notion of symmetry since it occurs frequently in objects they see every day. They have used line symmetry to cut out Valentine hearts with "sides that match" and snowflakes with symmetrical patterns. They have experienced that their feet are symmetrical rather than exactly the same when they tried to put the left shoe on the right foot. Investigation of instances of symmetry in real world objects should be a part of their study of symmetry. Geometrical shapes are used in this Investigation since they have more exact symmetry than objects in nature.

The terms used in this Investigation should be adapted to the level of the children. Most children find the term "line symmetry" more descriptive than the term "bilateral symmetry", and "turn symmetry" more descriptive than "rotational symmetry". As with most mathematical topics, symmetry is related to other topics. Seeing these relations strengthens understandings. For example, notions of symmetry can be investigated with the pentominoes by checking each for both line and turn symmetry (see Worksheet #1 of the Pentominoes Investigation).

As in all the investigations, the strategy of predicting the outcome of a particular action, and then checking the accuracy of the prediction by performing the action should be encouraged. It is urged that the children frequently be asked: "What do you think will happen if..."

Comments: Specific for the Investigation

1. The rectangle has two lines of symmetry, one horizontal and one vertical. When a figure has exactly two lines of symmetry, the lines are mutually perpendicular. Since the notion of perpendicularity develops later, younger children may not recognize this. Diagonals are not lines of symmetry since, when a rectangle is folded along a diagonal, the parts (though congruent) do not "match exactly, one on top of the other". Although symmetry implies congruency, congruency does not imply symmetry.

2. The equilateral triangle has three lines of symmetry. Each is an angle bisector, an altitude and a median. They intersect at the center of the triangle. In a figure that has 2 or more lines of symmetry, the lines intersect at a point which is the center for its turn symmetry (see #7).

3. The younger children may take some prodding to find all four lines of symmetry of the square. Since the square has 4 congruent sides, it has 4 lines of symmetry; the triangle had 3 congruent sides and 3 lines of symmetry. Challenge the students to predict, then check the number of lines of symmetry of a figure with 5 congruent sides, then with 6 congruent sides. Pose the question of how many lines of symmetry a circle has. Few can be expected to conclude that the circle has infinitely many lines of symmetry.

4. Most of the children are familiar with folding a piece of paper before cutting in order to have the parts match, e.g., making heart shapes. With one fold, one line of symmetry results. Challenge the students to find a way to fold so that two lines result (fold the paper into quarters before cutting).

5. If the children have never worked with mirrors before, allow time for many independent investigations to take place! For the younger children, the mirror test works best as a check for a line of symmetry which has already been found by folding. Initially, many children might try placing the mirror so that the whole figure is reflected. Remind them that the mirror is to be placed <u>on</u> the figure so that <u>half</u> of the figure is reflected, and half is in front of the mirror.

6. Emphasize that there are two tests for line symmetry: the fold test and the mirror test. The mirror test would be used on rigid figures that could not be folded.

7. The parallelogram will initially be puzzling, since the balance of parts suggests the parts should match (which they do, but by turning, not folding). Allow the children time to try various ways of folding before proceeding to the tracing and turning. A real world example of this process is fitting a top onto a box; whether or not it fits with part turns depends on the shape of the box. After re-examining the other shapes, some of the children will draw the generalization that if a figure has two or more lines of symmetry, it also has turn symmetry. Challence them to use this to make a design with turn symmetry by folding and cutting a piece of paper (fold it into quarters or eighths before cutting). Some children may have done this when making "snowflakes" in art. There is a lot of symmetry in art projects!

SYMMETRY

Investigation

1. Take a rectangle from the packet. Can you fold it in half so the parts match exactly, one on top of the other? A line on which you can fold a figure so the parts match exactly is called a line of symmetry. Can you find another line of symmetry of the rectangle? What do you notice about the two lines of symmetry of the rectangle? Why isn't the line from one corner to the other corner a line of symmetry?

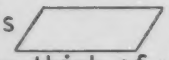
2. Now take a triangle from the packet. Are its sides the same length? How many lines of symmetry does it have? What do you notice about the lines of symmetry of this triangle? Do they meet at a point? Did the lines of symmetry of the rectangle meet at a point?

3. Now take a square from the packet. Before you fold it, predict how many lines of symmetry it has. Do you think it will have more than the rectangle? Why? Now try folding it. How many lines of symmetry did you find? What test do you make to find out if a line is a line of symmetry?

4. Now make your own figure that has a line of symmetry. Take a piece of paper, fold it in half, and cut it any way you like, beginning at one fold corner and ending at the other. Open it up and look at it. How do you know it has line symmetry? Trade figures with a friend, and test each other's for line symmetry.

5. Take a mirror from the packet. Place the edge of the mirror on the line of symmetry of your figure. What do you see? The mirror gives another way of testing to see if a line is a line of symmetry. If the mirror can be placed on a line so the whole figure can be seen, half reflected in the mirror and half in front of the mirror, then the line is a line of symmetry.

6. Do the lines of symmetry you found on the square and on the rectangle and on the triangle check with the mirror? When would the mirror test be better than the fold test for a line of symmetry? When would the fold test be better?

7. Take the shape that looks like this  from the packet. It is called a parallelogram. (Can you think of why that is a good name for it?) Using either the fold test or the mirror test, see if you can find a line of symmetry for this shape. Does it have a line of symmetry? Now trace around the parallelogram on a piece of paper. Mark the top with an "X": Can you turn the shape part of a full turn and have it match its tracing exactly? Does it match with a half turn, when the "X" is at the bottom? If a shape can be turned part of a full turn and look just as it did before, then it has <u>turn symmetry</u>. So this shape, the parallelogram, has turn symmetry. Which of the other shapes (rectangle, triangle, square, one you made) has turn symmetry? How do you know?

8. Encourage applying symmetry notions to environmental objects. Even the younger children will readily recognize that their own bodies exemplify symmetry. Some children may want to make a collection of natural shapes which are symmetrical, e.g., leaves, butterflies, wheels. The objects of nature will only approximate symmetry.

9. "Symmetry" is used in both terms since "symmetry" means a balance of parts, either across a line or around a point.

8. Look around you. Name three things which have line symmetry and three things which have turn symmetry. Do some have both? What do you notice about the part turns of a figure with turn symmetry and the number of lines of symmetry it has?

9. Why do you think the word "symmetry" is used for both line symmetry and turn symmetry?

SYMMETRY

Worksheet #1

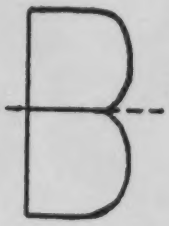
1. Here is an alphabet printed in capital letters:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Look at symmetry. It is "balanced" about the dotted line, the line of

Write down all the other letters which appear balanced about an "up and down" line of symmetry. Show the line by a dotted line.

H I M O T U V W X Y

2. Look at symmetry.  It is balanced about the dotted line, the line of

Write down all the other letters which appear balanced about an "across the page" line of symmetry. Show the line by a dotted line.

B C D E H I K O X

3. Which letters have both "up and down" lines of symmetry and "across the page" lines of symmetry? H, I, O, X

4. Can you find a word that has a line of symmetry? Does BOB have an "across the page" line of symmetry? What kind of symmetry does TOOT show if you fold it between the O's? T O O T has symmetry if the whole word is considered.

5. This letter looks the same if you turn it through half a complete turn.

Find all the other letters which look the same when turned in this way. They have turn symmetry.

H, I, O, S, X, Z

SYMMETRY

Worksheet #1

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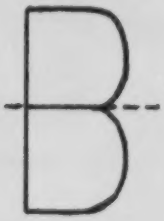
2. Look at symmetry.  It is balanced about the dotted line, the line of

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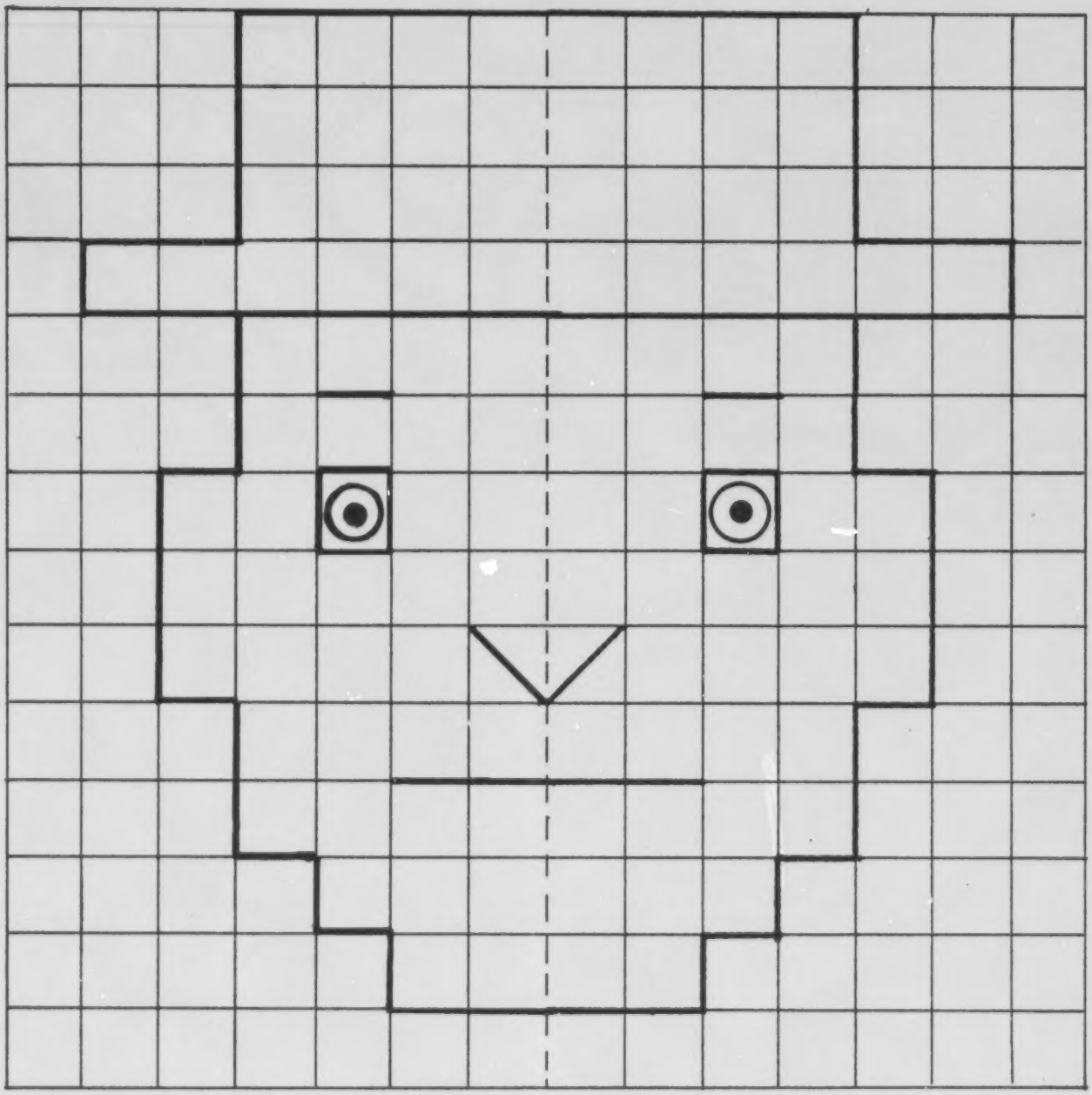
Find all the other letters which look the same when turned in this way. They have turn symmetry.

SYMMETRY

Worksheet #2

Complete the picture of the person's face so that it has line symmetry about the dotted line.
How can the squares of the grid help you to find where to draw your lines?
What other tests can you make to find out if you have placed your lines correctly?



SYMMETRY

Worksheet #2

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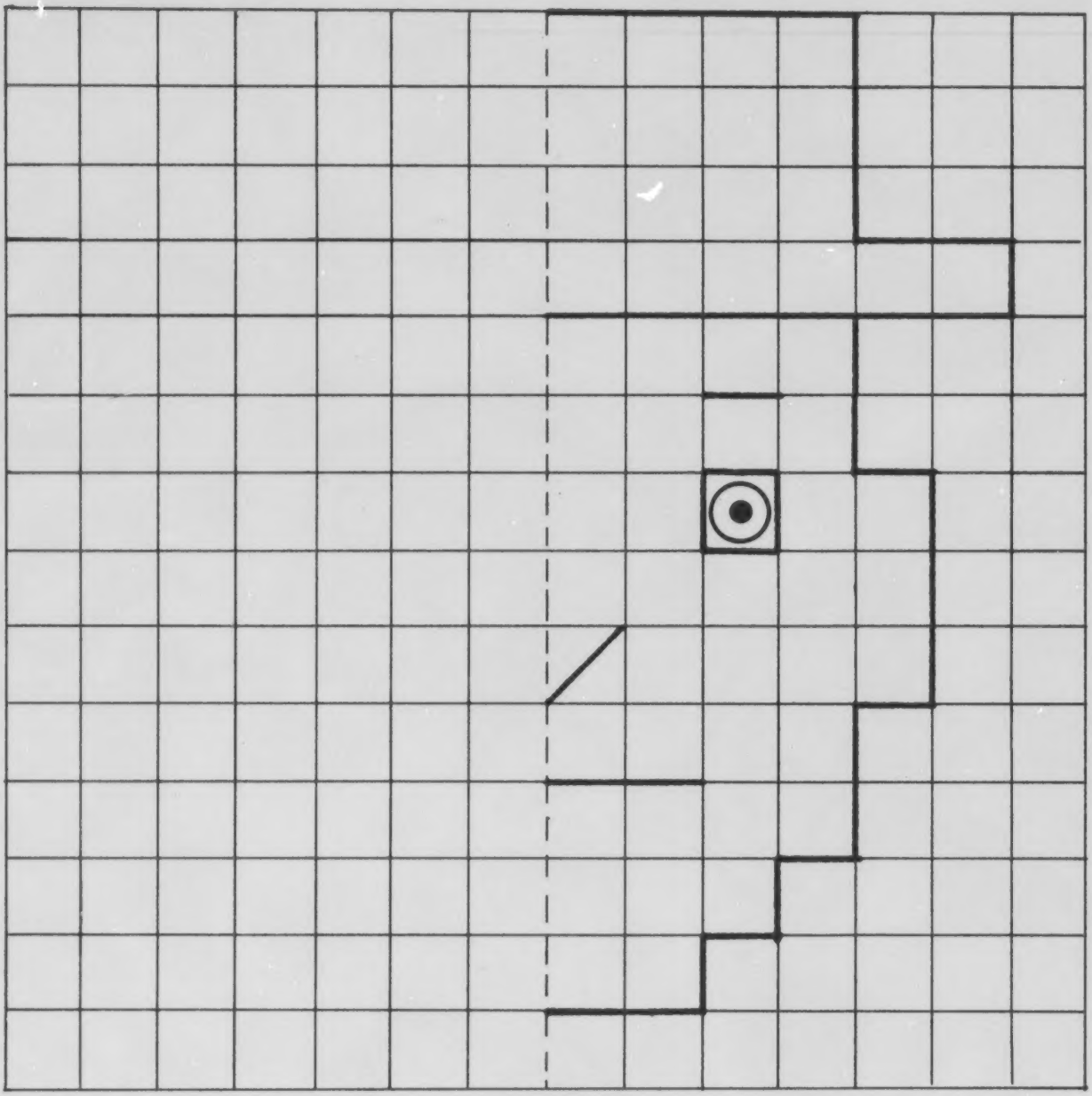


TANGRAMS

Notes to the Instructor

Objectives: to discover and reinforce geometric properties through manipulations of tangrams;
to provide practice in spatial perception;
to form specific shapes given the component parts;
to recognize and form congruent and similar figures;
to reinforce the concept of area as a measure of covering, and to find areas of given shapes by counting units;
to compare perimeters of figures whose areas are equal;
to test figures for rotational and bilateral symmetry.

Materials: Set of tangram pieces: purchased, or made from construction paper and laminated; one set per student.
Unlined paper: three sheets per student.
Pencils.
Scissors.
Worksheet: one per student.

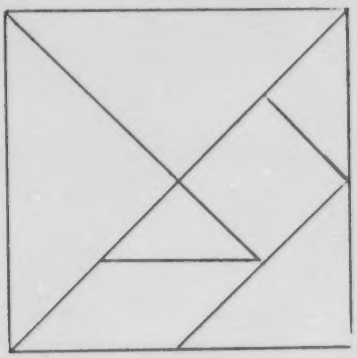
References: "Classroom Proven Motivational Mathematics Games" Monograph No. 1 of the Michigan Council of Teachers of Mathematics. December, 1971: 27-29.

Seymour, Dale. Tangramath. Palo Alto, California: Creative Publications, 1971.

Sobel, Max A. and Maletsky, Evan. Teaching Mathematics: A Sourcebook of Aids, Activities, and Strategies. Englewood Cliffs, N.Y., Prentice-Hall, Inc., 1975: 78-79.

Comments: General

The tangram pieces are perhaps the oldest enduring geometric puzzle. The 7 pieces are: two large triangles, a medium triangle, two small triangles, a square, and a parallelogram. These pieces are of specific proportions, as seen by the way they are formed from a square. If time permits, much can be gained by having the children make their own tangrams from an 8" x 8" square of lightweight cardboard, such as a manila file folder. Use this pattern:



Allow plenty of time for the children to manipulate the pieces making their own geometric and pictorial designs before challenging them to make specific arrangements. This permits them to find the relationships among the pieces, and can help prevent frustrations which will result from challenges they are not ready for.

For the younger children, pattern cards should be made by outlining the required design large enough so the pieces fit directly on the outline, as done on the Worksheet.

Designs made from the tangrams make a colorful bulletin board display. Have the children add to the display as they find new designs.

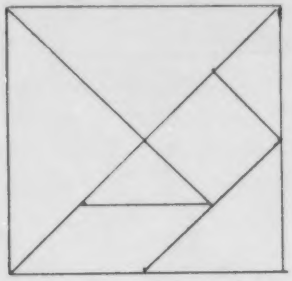
Activities with tangrams can be adapted for a wide range of age and ability. The first several problems are appropriate for young children, while the last few should be used in intermediate grades. The younger children enjoy and benefit from informal explorations fitting the pieces together to form designs. Older children extend this to designs using specific pieces forming specific polygons, and can then be asked about the areas and perimeters of these polygons.

The emphasis in this Investigation is on finding areas and perimeters by counting. A natural extension is to discover the area formulas by comparing the area to the dimensions of rectangles and triangles made from the tangram pieces.

Comments: Specific for the Investigation

1. Initial manipulations should permit discovery of many of the interrelationships of the pieces. Use the term "congruent" for the same shape-size relationship illustrated by the two small triangles. An appropriate test for congruent shapes is that one piece fits on top of the other. Use the term "similar" for the same shape-different size relationship illustrated by the medium and a large triangle.

2. The pairs of congruent triangles form squares, illustrating that the diagonal of a square divides it into congruent right triangles. Encourage students to also form squares using 3 pieces, then 4 pieces, 5 pieces, all 7 pieces. Many tangram activities begin by making the pieces form a square:

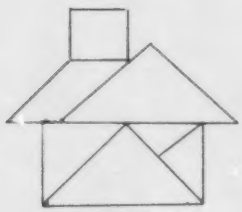


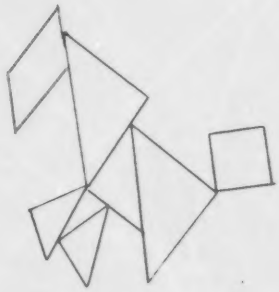
3. The concept of area as the number of units it takes to cover is illustrated here. Also reinforced is conservation of area: since the two small triangles can be made to cover first the square and then the parallelogram, the latter are equal in area.

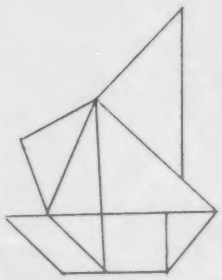
4. Each of the areas found in the preceding problem can be multiplied by 2, or found by direct comparison. The square is 2, the parallelogram 2, medium triangle 2, large triangle 4.

5. Finding an arrangement to fit a given outline can be made easier by giving the positions of one or two key pieces, or by making the outline the same scale as the tangram pieces.

6. The more compact the design, the more difficult it is for children to make. Challenge the fast students to make squares, rectangles, parallelograms, etc.







TANGRAMS

Investigation

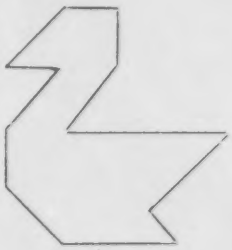
1. Take a set of 7 tangrams from the packet. What shapes are in the set?
   What do you notice about how the pieces compare with each other?
   Do some of the pieces match in size and shape?
   Do some match in shape but not size?
   Are the sides of some of the different shapes the same length?
   How can you tell?

2. Using only two pieces at a time, can you make a square?
   Can you make another square with another pair of pieces?
   Is one of these the same size as the square piece? How do you know?
   Can you make a triangle using two pieces?
   What can you say about this triangle?

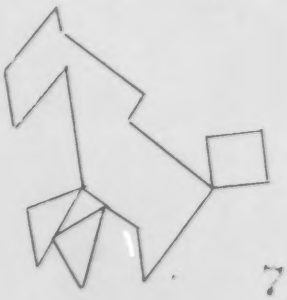
3. Suppose the square piece has area 1. What is the area of each small triangle? Use the small triangles to find the areas of the medium triangle, and of the parallelogram. What is the area of the large triangle? Can you show this two ways? How do the areas of the square, the parallelogram, and the medium triangle compare? Can different shapes have the same area?

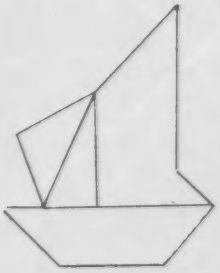
4. Now suppose the area of the small triangle is 1. What is the area of each of the other pieces? How do you know? If the large triangle has the area 2, what piece has area 1?

5. Make a design using five pieces. If the area of the square piece is 1, what is the area of your design? Now make a design using all seven pieces. What is the area of that design?
   Can you make this swan using all seven pieces? (Hint: the parallelogram is the neck and the wing is a large triangle.) What is the area of the swan?



6. Here are some more designs; each uses all seven pieces. Which are easy to make? Now you make up a design, trace around it to make an outline, then see if a friend can make your design from just the outline.





7. The areas of the arrangements are equal since the same pieces are used, but the perimeters vary; a constant area does not imply a constant perimeter. The polygon with the smallest perimeter for a fixed or given area is always a square. Perimeter may be found by measuring or by applying the Pythagorean Theorem. This exercise can be extended to using 5 or using all 7 pieces to form a square, a rectangle that is not a square, a triangle, a trapezoid and a parallelogram, and comparing the areas and perimeters of each. The students should find that although the area remains constant, the perimeter can vary; the square has the smallest perimeter.

8. A figure which matches itself when flipped has line or bilateral symmetry; a figure that matches itself with part of a full turn has turn or rotational symmetry. (See the Symmetry Investigation.) The triangles have line but not turn symmetry; the parallelogram has turn but not line symmetry; the square has both.

9. A "fit" or "match" means congruent parts are involved. All three triangles are isosceles right triangles, and so have congruent base angles and congruent legs. A similar investigation of congruent parts of the parallelogram is contained in the Worksheet. In the same way, show the square has 4 congruent sides and 4 congruent angles.

7. For this problem let the square have area 1. Using just the two small triangles, make a square. What is its area? What is its perimeter? Using the same two pieces, make a triangle. What is its area? What is its perimeter? Using the same two pieces, make a parallelogram. What is its area? What is its perimeter? Make a statement about how the areas and perimeters of these compare.

8. Trace around the medium triangle. Does the triangle fit its outline if you flip it over? Will it fit its outline if you turn it part of a full turn? Which other pieces do you think will fit their outlines when flipped? Which will fit their outlines if turned part of a full turn? Check your predictions.

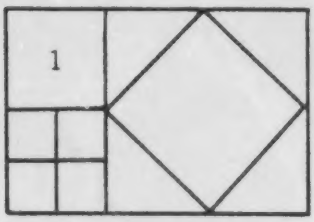
9. When a piece fits its outline after a flip or a turn, what does that tell you about the parts of the piece? Each of the triangles fits its outline after a flip; how does this show you that two of the angles of the triangle are the same size? A triangle with two congruent (same size) angles is called isosceles. Use the square piece to "measure" the third angle of a triangle piece. What is the name of this kind of angle? How does it compare in size to the two equal angles of the triangle? Now use the flip-and-match to make a statement about congruent sides of an isosceles triangle.

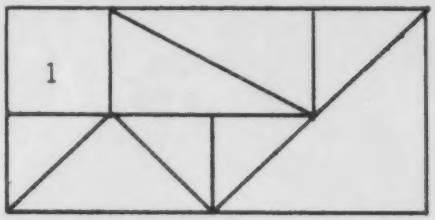
Worksheet

TANGRAMS

1. This square has area 1:

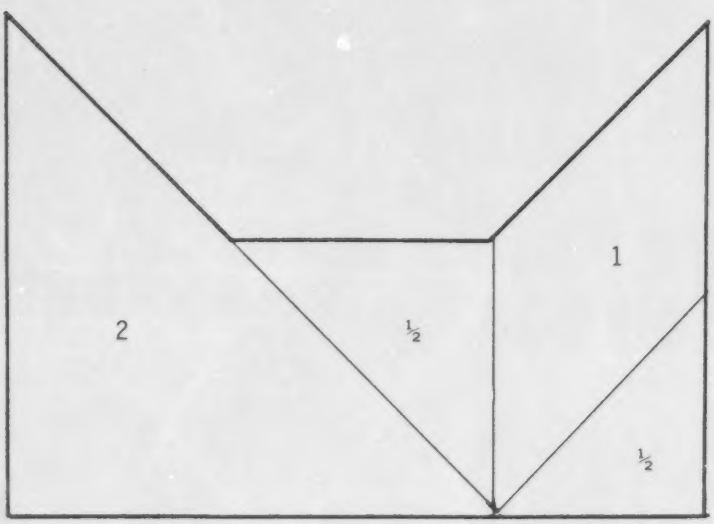
Label the area of each part of the rectangle, and find the total area of each rectangle:

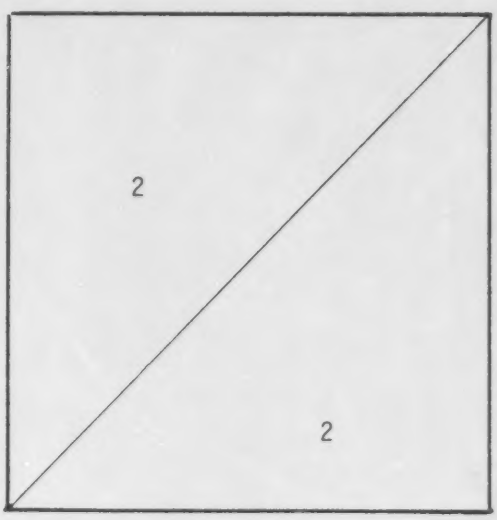


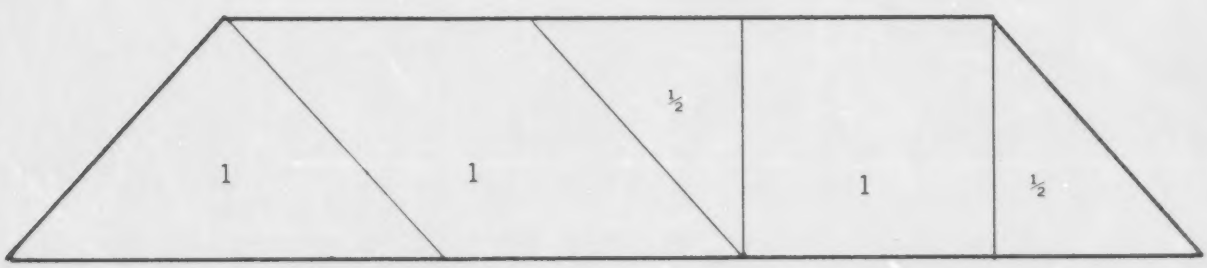


If the side of the unit square is 1, what is the perimeter of each rectangle?

2. Which of these shapes do you think has the largest area? Now use your tangrams to find the area of each. Were you right?





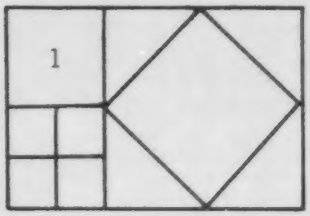


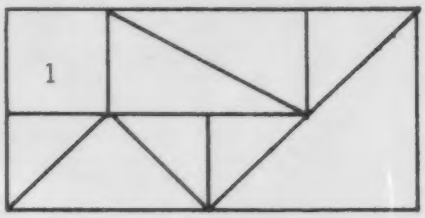
Worksheet

TANGRAMS

1. This square has area 1:

Label the area of each part of the rectangle, and find the total area of each rectangle:





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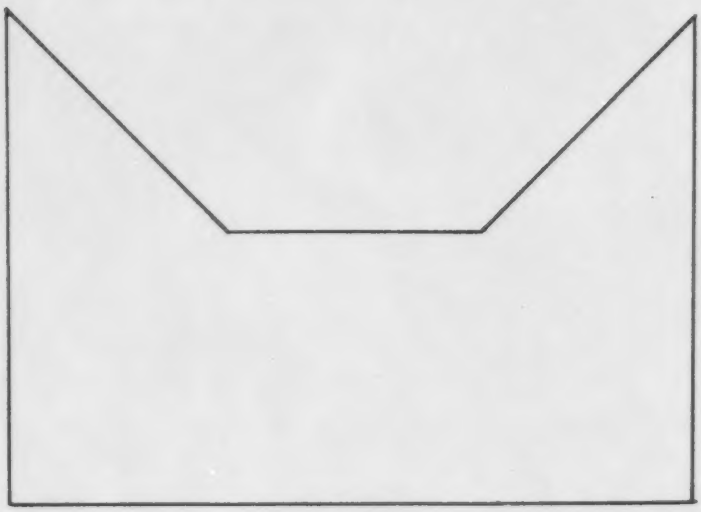
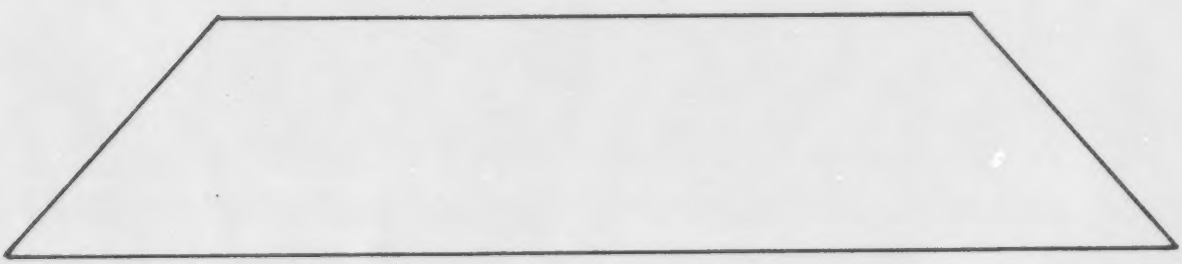
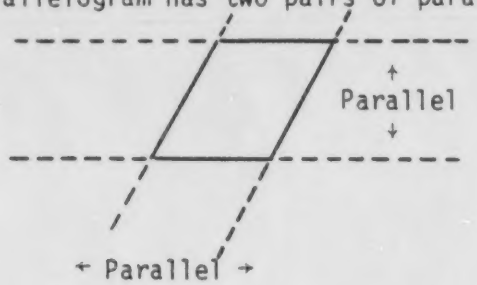
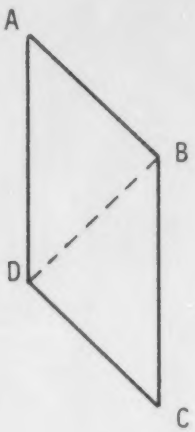




3. A parallelogram has two pairs of parallel sides:



What else is true about all parallelograms?



a) Use a small triangle to compare the lengths of the sides of the parallelogram. Are the opposite sides of a parallelogram congruent?

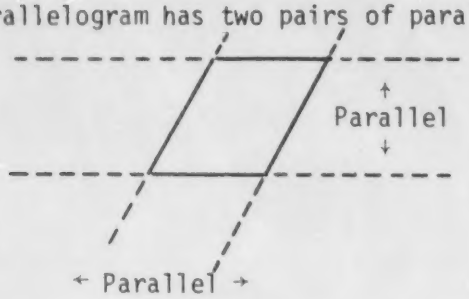
b) Fit the parallelogram into the outline, then flip it to show that angle A is congruent to angle C, and flip it another way to show angle D is congruent to angle B. Did you find that the opposite angles of the parallelogram are congruent?

Are these statements true for all parallelograms? Use your tangram pieces on the parallelograms below to show:

opposite sides are congruent
opposite angles are congruent

As an extension, have the students find the areas and perimeters in terms of the square piece, as shown on the key. They may also use the tangrams to verify the formula A = bh.

3. A parallelogram has two pairs of parallel sides:



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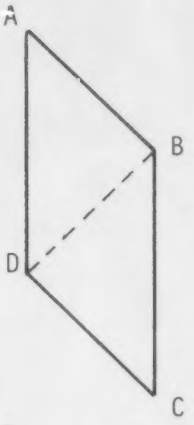

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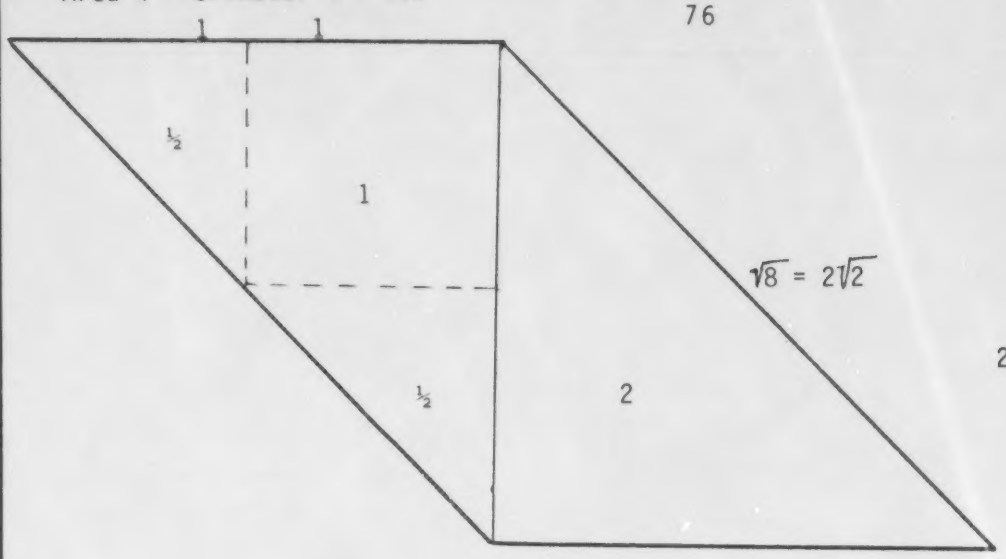
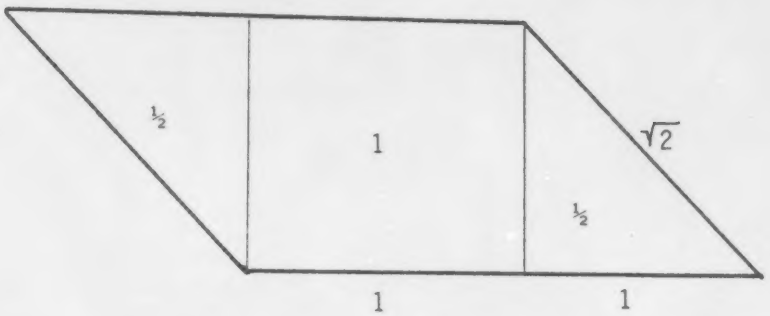
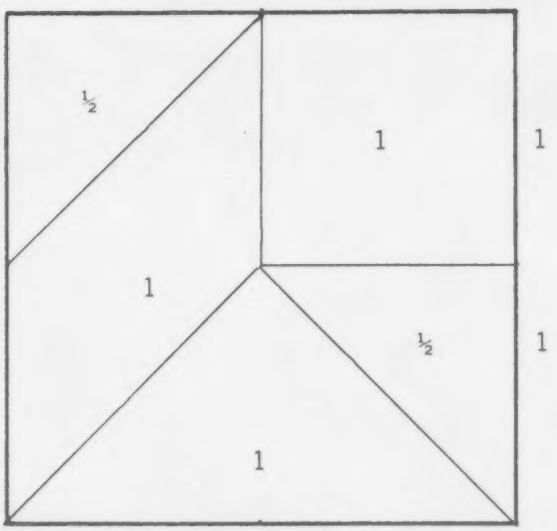
Area 4 Perimeter $4 + 4\sqrt{2}$



Area 2 Perimeter $4 + 2\sqrt{2}$

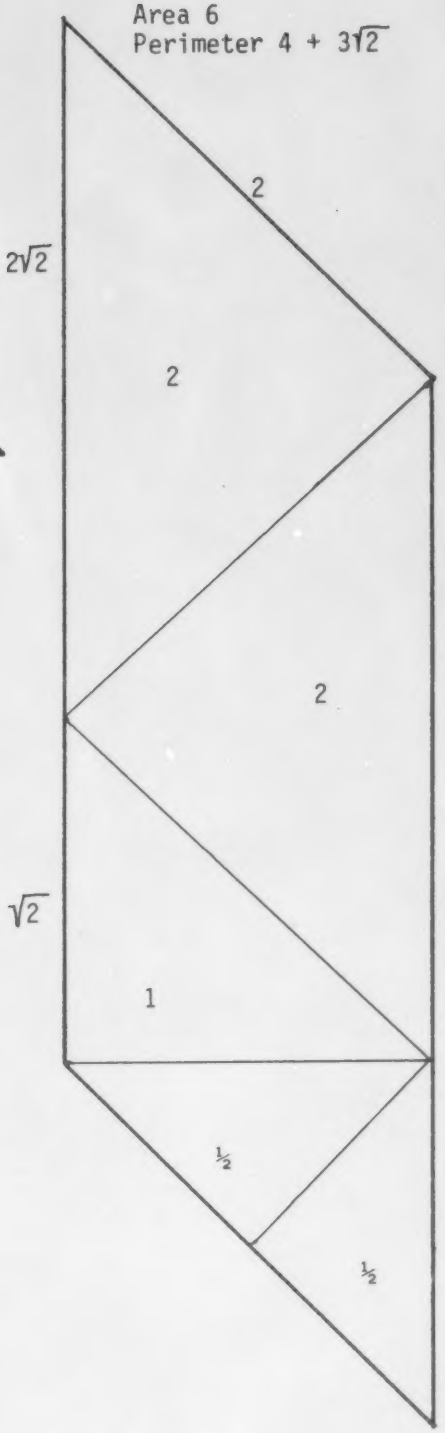


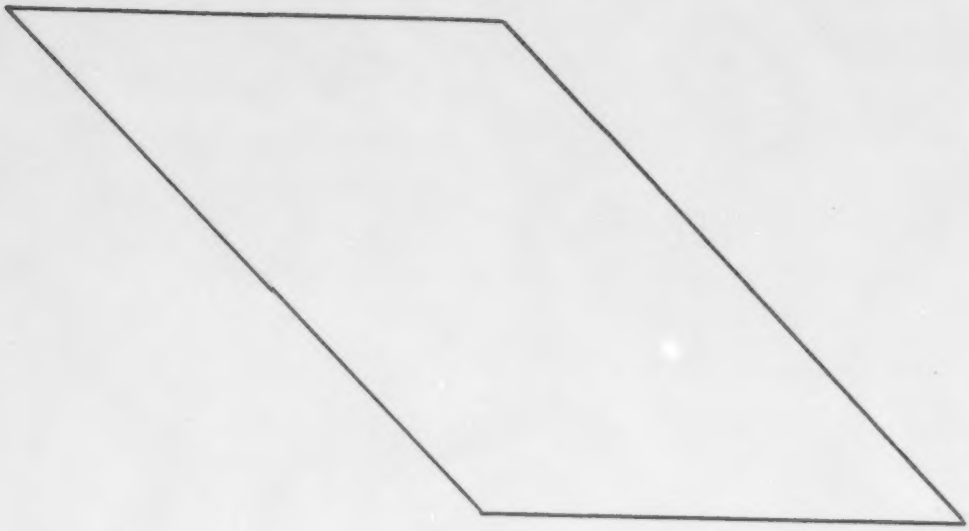
Area 4 Perimeter 8

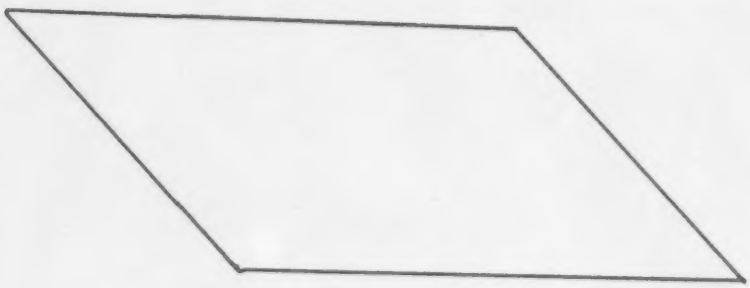


(Are squares parallelograms?) Yes

Area 6
Perimeter $4 + 3\sqrt{2}$







(Are squares parallelograms?)

TILING

Notes to the Instructor

Objectives: to find that certain polygons can be used to tile (cover without gaps or overlaps) the plane while other polygons cannot;
to recognize the use of tiling patterns in the environment;
to use tiling patterns to form artistic designs.

Materials: Polygons cut from colored construction paper, 10 copies of each:
equilateral triangles, 3 cm on a side, red;
regular hexagons, 3 cm on a side, brown;
regular pentagons, 3 cm on a side, purple;
parallelograms, any size, dark green;
trapezoids, any size, yellow;
"kite" quadrilateral: ◇ any size, blue;
squares, 3 cm on a side, white;
regular octagons, 3 cm on a side, orange;
regular dodecahedron, 3 cm on a side, light green.

References:

Association of Teachers of Mathematics. Notes on Mathematics in Primary Schools. Chapter 7, "Tessellations". New York: Cambridge University Press, 1968.

Bolster, L. Carey. "Activities: Tessellations." Mathematics Teacher 66:147-161, April 1973.

Clemens, Stanley. "Tessellations of Pentagons." Mathematics Teaching 67. June 1974.

Escher, M. C. The Graphic Work of M. C. Escher. New York: Hawthorne Publishing, 1966.

Mold, Josephine. Topics from Mathematics--Tessellations. New York: Cambridge University Press, 1969.

Monie, H. M. J. "Tessellating Hexagons." Mathematics Teaching. 54:26-27, Spring 1971.

Nuffield Mathematics Project. Shape and Size (No. 2 & 3). New York: John Wiley & Sons, 1968.

O'Daffer, Phares and Clemens, Stanley. Geometry: An Investigative Approach. Chapter 3: "Patterns of Polygons in the Plane". Menlo Park, California: Addison-Wesley, 1977.

Comments: General

This investigation involves finding patterns formed by fitting polygons together to cover the plane. These patterns are a form of art that has been used for floor and wall coverings and fabric designs since the times of the early Greeks, Egyptians and Chinese. A more complex form of this art is seen in the works of M. C. Escher (1898-1972), a Dutch painter,

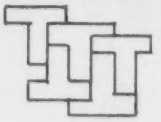
whose popular works display the sophistication to which this procedure of fitting together congruent shapes can be developed (see reference).

Work with these patterns, called "tilings" because the connection with tiling floors, or "tessellations", provides a natural avenue for integrating geometry and art, and helps students see math as an enjoyable activity.

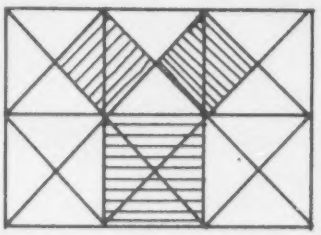
The idea of an infinite repetition of pattern can be developed in younger children by describing how it can go on and on, covering the whole floor, past the walls, and even cover the whole playground; a flat space, no matter how big, could be covered.

Although the Investigation can be done on practically a non-verbal level, the wording can be made less awkward if students know the terms such as equilateral triangle, polygon, congruent, etc. Using color-coded shapes is helpful if students do not know the names of the polygons.

Tiling connects easily with other areas of study. For example, each of the pentominoes can be used to make a tile pattern:

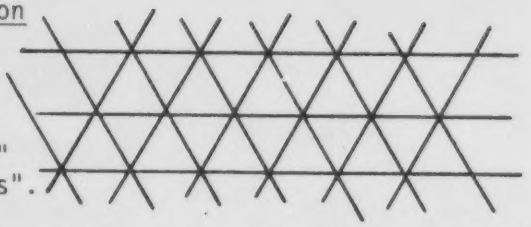


Also, a tile pattern of right triangles can be used to discover a special case of the Pythagorean Theorem:

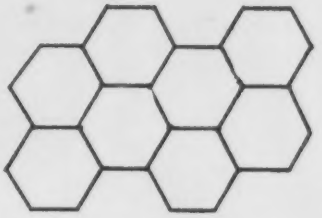


This Investigation should be followed by providing students with opportunities to develop their own tiling patterns, as well as to find examples of tilings in their environment, such as floorings, wallpaper, brick walls, in nature, etc.

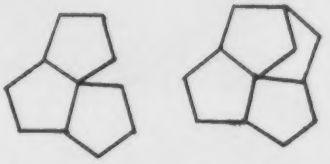
Comments: Specific for the Investigation



1. The pattern is readily found: A "tiling pattern" can be extended infinitely to cover a plane surface without gaps or overlaps. "Tilings" are also often called "tessellations". Many shapes can be found in this tiling: parallelograms, rhombuses, hexagons, trapezoids, larger equilateral triangles.



2. The tiling of regular hexagons is the familiar honey-comb pattern. If symmetry has been studied, investigate tiling patterns for both line and turn symmetry. This pattern, for example, has both kinds of symmetry.



3. The only regular polygons which tile the plane are equilateral triangles, squares, and regular hexagons. Three regular pentagons surrounding a point leave a gap, while four overlap: It takes 360° to surround a point. Each angle of a regular pentagon is 108°. When three are put together, the sum is 324°, less than the 360°; when four are put together the sum is 432°, more than the 360° needed. The angles of the regular polygons that tile are factors of 360°.

4. Any quadrilateral will tile the plane. Encourage faster students to show this with concave quadrilaterals, too. One method for finding the pattern especially helpful for irregular quadrilaterals, is to successively turn it about the midpoint of each side, thus generating four copies of the quadrilateral about a point. Translations of this pattern, made with tracing paper, then easily show the tiling pattern.

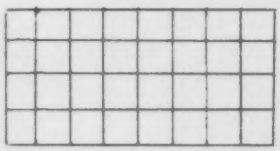
As the quadrilateral is turned about first M_1, then M_2, then M_3, the position of angle A is A', then A", then A"'. A slide of this pattern extends it in four directions infinitely, tiling the plane.

Any scalene triangle tiles the plane, since any triangle can be paired with a copy of itself to form a parallelogram (a special quadrilateral), and it has just been shown that any quadrilateral tiles the plane. By noting how

TILING

Investigation

1. A square can be used to tile a floor because same-sized square tiles can be fit together leaving no holes and without making overlaps. The pattern the squares make can go on and on, covering any size floor.



Do you think an equilateral triangle could be used to tile a floor? Take 10 equilateral triangles (red shapes) and fit them together to show a tiling pattern. Be sure to have the sides match completely, corners touching like this:

not like this:

2. Now take 10 of the hexagons (brown shapes). Can you make a tiling pattern with them? Have you seen this pattern before in nature?

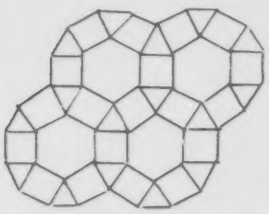
3. The shapes used so far have been special: their sides were the same length. They are called regular polygons. Do you think any regular polygon will make a tiling pattern? Take some regular pentagons (purple shapes) and try to make a tiling pattern. What do you notice?

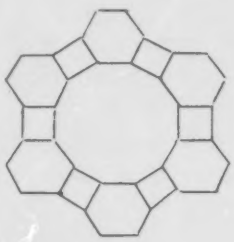
4. Will shapes without equal sides make a tiling pattern? Take 10 parallelograms (green shapes). Can you make a tiling pattern with them? Try to make a pattern with 10 trapezoids (yellow shapes). Now try the "kite" shape (blue shapes). Do you think any 4-sided shape will make a tile pattern? If you are not sure, draw a 4-sided shape, cut it out, make 10 copies of it, and see if you can find a pattern.

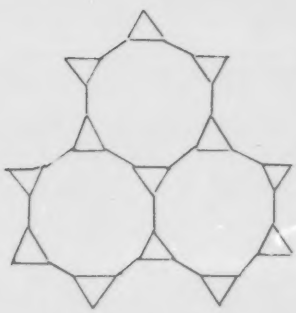
Will any triangle make a tiling pattern? Draw any triangle you like (not too small to work with) and cut out 10 copies of it. Try to fit your triangles together to make a repeating tiling pattern. Do the triangles fit together in pairs to make same-shaped (congruent) 4-sided polygons? What did you find out about such polygons in the first part of this question?

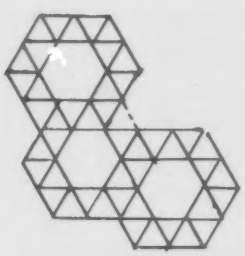
the angles of a scalene triangle fit together to form a tiling pattern students can discover that it takes 2 copies of each of the 3 angles to fit about a point, i.e., $2(a+b+c) = 360^{\circ}$ so $a+b+c = 180^{\circ}$ where a, b, and c are the measures of the 3 angles. Some students may want to try to find tiling patterns with other irregular polygons. An interesting extension is to find the tilings of the pentominoes (see Pentominoes Investigation).

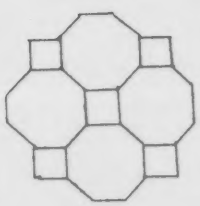
5. Combinations of two or more regular polygons form exactly 8 semi-regular tiling patterns. Notice that to form a pattern, the sum of the angles meeting at a point must equal 360°. For example, two squares and three triangles can surround a point since $90^{\circ}+90^{\circ}+60^{\circ}+60^{\circ}=360^{\circ}$.

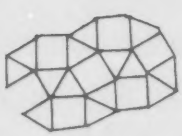


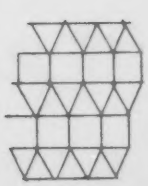












5. Can tiling patterns be made with combinations of regular (having equal sides) polygons?

   a. Use the squares and equilateral triangles to make a tiling pattern. Why does it work?

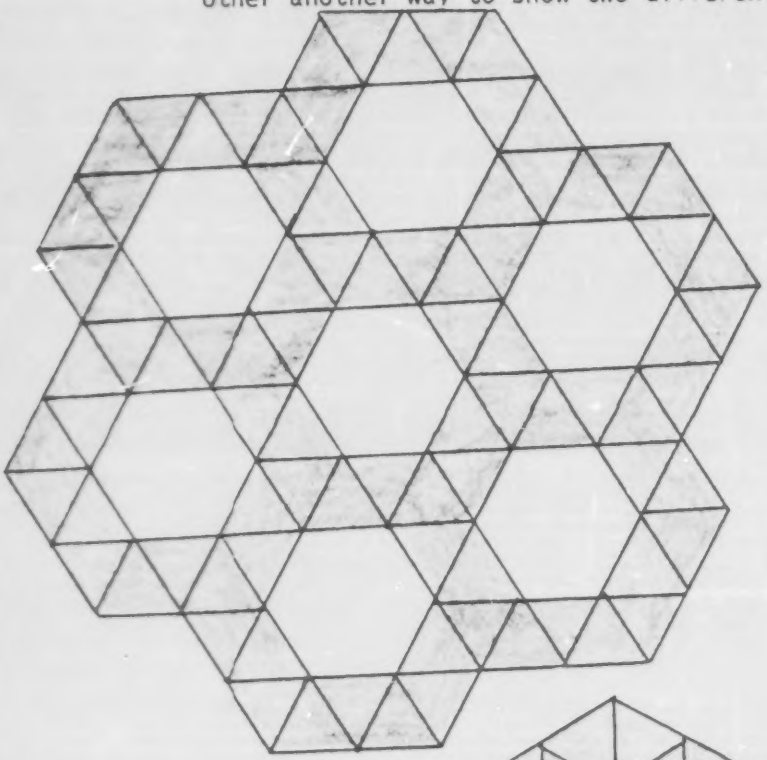
   b. Use the squares and the 8-sided polygons (octagons) to make a tiling pattern.

   c. Can you find other combinations of the regular polygons that fit together in a tiling pattern?

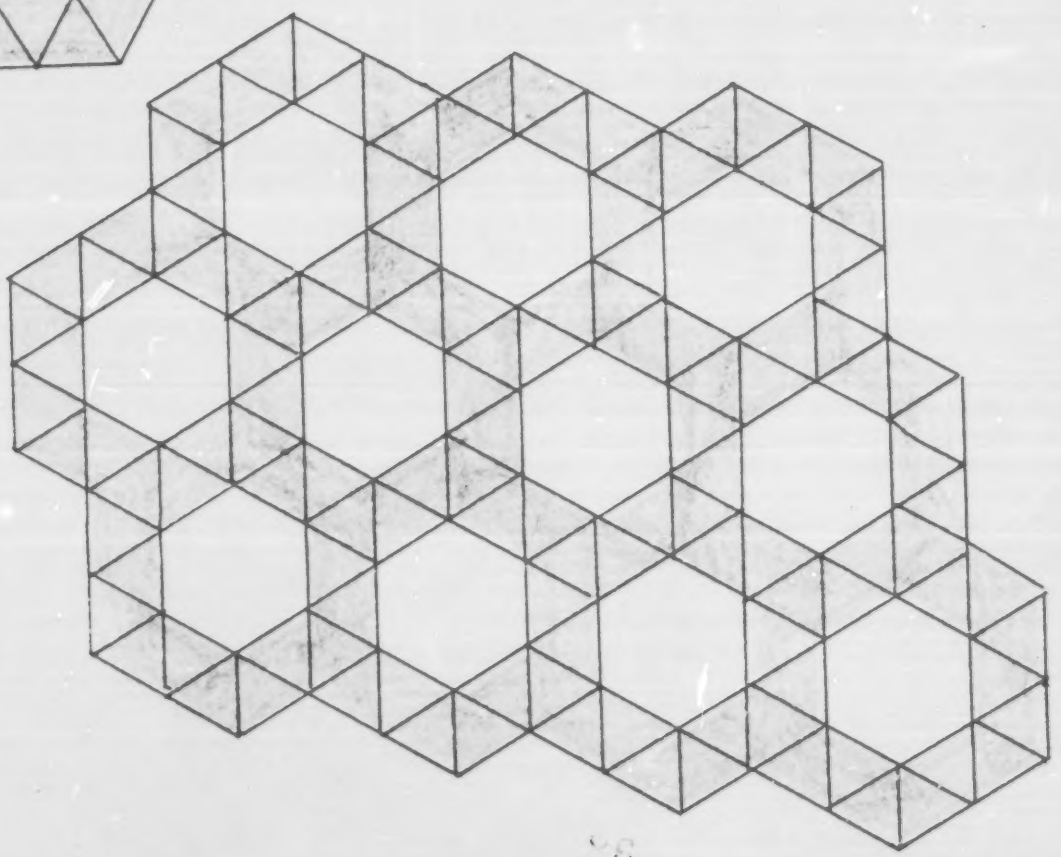
TILING

Worksheet #1

Here are two copies of a tiling pattern. Notice that they are the same pattern. One is a part turn of the other. Color one one way and the other another way to show two different patterns within the pattern.



By orienting a tiling pattern differently, different patterns within the pattern are seen. Using different shades of the same color can emphasize the overlapping design.



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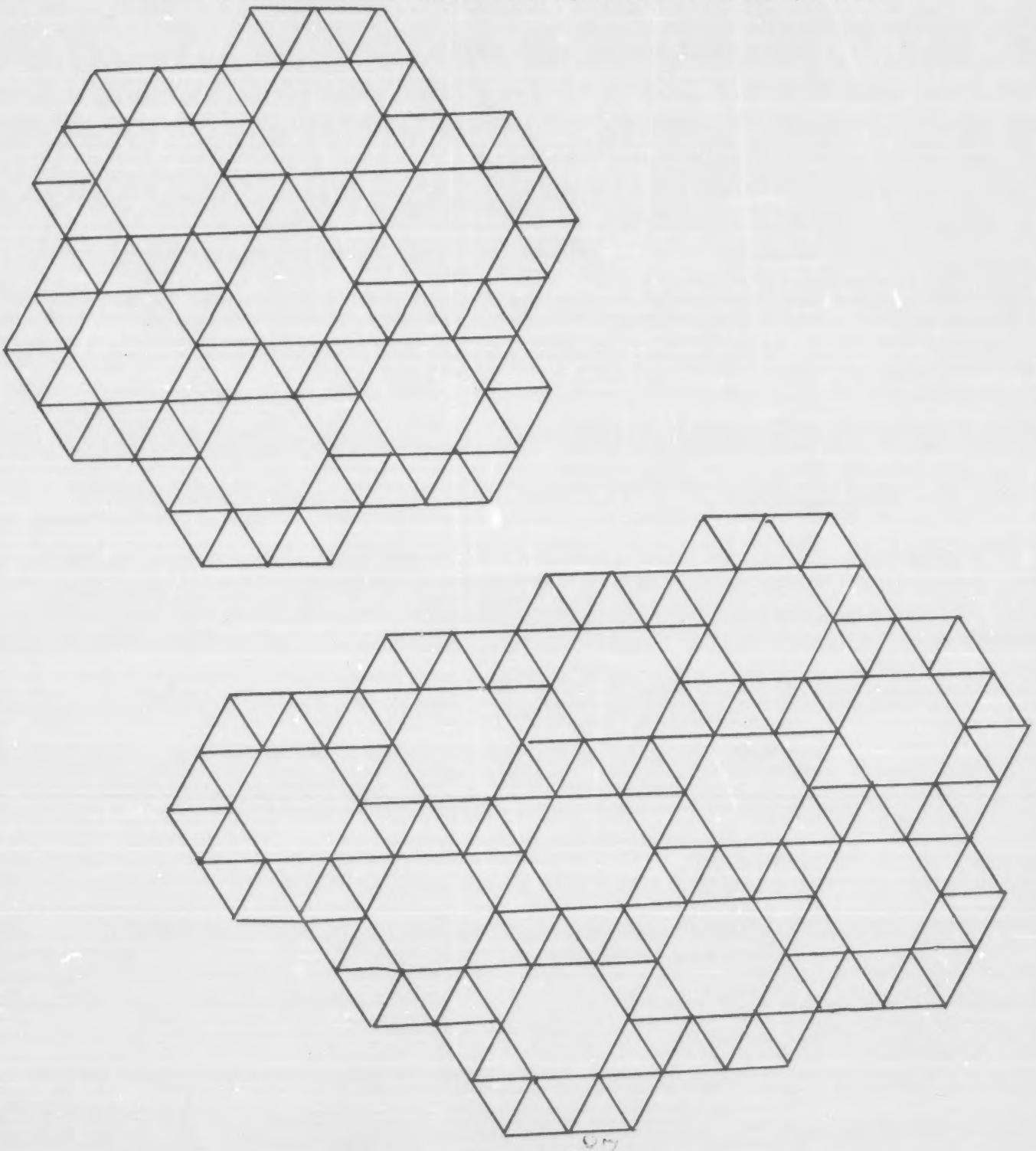
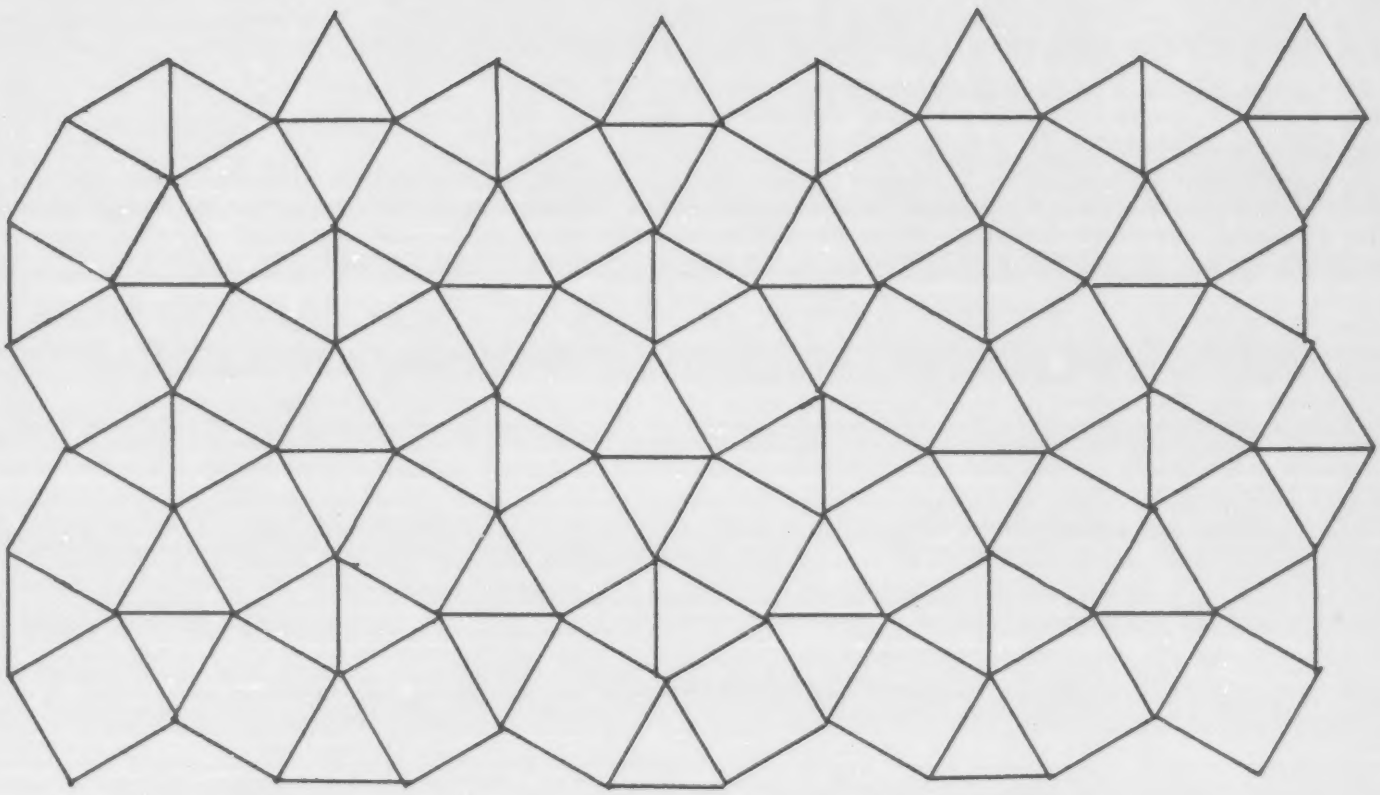


TILING

Worksheet #2

Color this tiling of squares and equilateral triangles to show an interesting design.

Many different designs are possible by coloring different sets of adjacent polygons alike.



TILING

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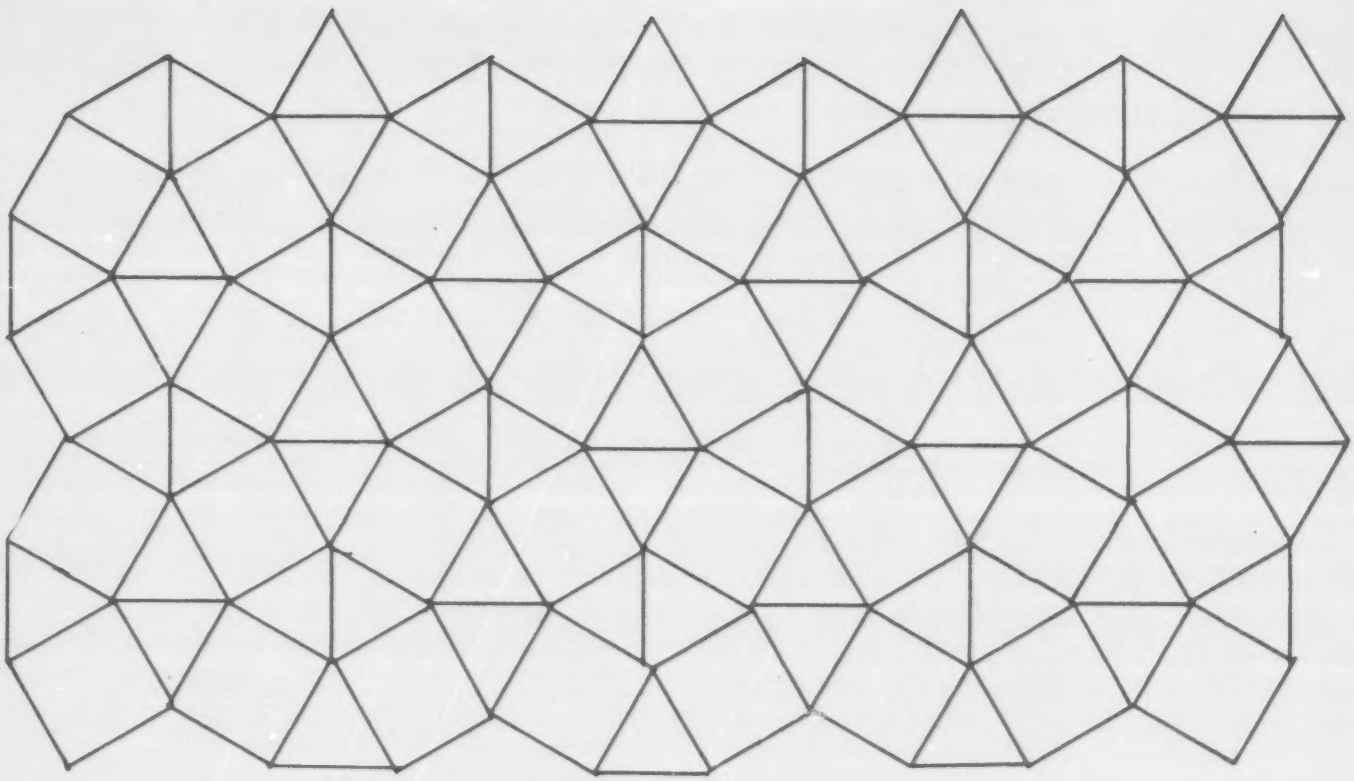


TOPOLOGY

Notes to the Instructor

Objectives: to introduce the area of mathematics called topology by investigating some of the unusual properties of this area;
to investigate the properties of the Möbius Band;
to develop the idea of a network and investigate traceability of networks.

Materials: Paper strips: 1" x 14" (cut from adding machine tape) - 8 per student.
Pencils: one per student.
Glue: one per four or five students.
Scissors: one per student.
Crayons: two different colors per student.

References: Bergamini, David et al. Mathematics (Life Science Library). New York: Time, Inc. (1963): 176-189.

Johnson, Donovan A. "Topology" in Enrichment Mathematics for the Grades. 27th Yearbook of National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1963.

Nuffield Mathematics Project: Problems - Purple Set. New York: John Wiley and Sons, (1972): 3-4, 31-33, 77-78.

O'Daffer, Phares G. and Stanley R. Clemens. Geometry: An Investigative Approach. Menlo Park, California: Addison Wesley, (1976): 259-282.

Comments: General

Topology is a branch of geometry that deals with properties of positions that are not affected by changes in size and shape. Topological notions such as inside, outside, and between are encountered by even the youngest children. In fact, researchers, such as Piaget, suggest that topological notions are developed by children before Euclidean notions. For example, very young children often have difficulty differentiating a circle and a square, which are topologically the same: both are simple closed curves, dividing the rest of the plane into exactly two parts, inside and outside. Classifying shapes which are topologically the same is an appropriate extension to this lesson. A concrete way of doing this is to draw the shape on a piece of burst balloon and seeing what shapes it can be distorted into by stretching or bending, but not tearing the rubber. Because topology is interested in properties which are unchanged by this type of action, it is nicknamed "rubber sheet geometry."

The Möbius Band and networks were chosen for investigation because of their general intrigue, for both young and old, and because they are better known applications of topology. Networks are used by electricians to show the relative position of switches on circuits, by individuals drawing rough sketches to give travel directions, by transportation people to schematically show stops and connections, and by many other companies and individuals as a way of isolating the relevant facts related to position.

The Konigsberg Bridge Problem is one of the famous problems of mathematics. It may be attempted on a trial and error level by the youngest students, and then, by introducing a minimum number of concepts, may be solved readily by mathematics. It is a topological problem since it is concerned with the position of points (betweenness) rather than distance or shape (Euclidean notions) of the paths connecting the points. To appreciate the power of the notions introduced on the worksheet in enabling the solution of the problem, the children should first attempt to solve the problem by their own means, usually trial and error.

Comments: Specific for the Investigation

1. In making the Möbius (or Moebius) Band, be sure the students have given the strip exactly one half twist before glueing; a full twist is not a Möbius Band. The glue should extend to the edge of the paper so the loop will not come undone when cut in the following steps.

2. Much of the value of this investigation comes from the unpredicted, surprising results. Therefore, be sure to have each child say what he thinks will happen before cutting the loops in half lengthwise. Most will guess two loops will result. Ask them what the two loops will be like; will they be able to put one on one arm, the other on the other arm, or will they be looped together? Then let them cut. The result is one large loop, with a full twist in it.* When this loop is cut in half, the result is two intertwined loops (what was expected the first time!) since this was not a Möbius Band. The mathematicians' explanation: A Möbius Band has but one edge: the cut adds a second edge--and a second side.

3. Again, have the children predict the outcome before cutting. After the previous result, many will guess that a loop three times as big will result, since the cut is one-third in from the edge. If some guess two loops, ask them about the relative size of the loops, their width and length. The result is two intertwined loops, one longer two-sided loop, and one small Möbius Band. While making this continuous cut, make sure the children cut one-third in from the edge and do not run off the edge.

4. Coloring both sides of an ordinary paper loop poses no problems, but coloring a Möbius Band proves that it has only one side. The loop in step #2 was two-sided, and could be colored with two colors. The large loop from #3 was two-sided, but the smaller had only one side.

5. & 6. Continue investigating until the children see that a Möbius Band is the result of half twists. An even number of half twists gives full twists and the band is two-sided, not Möbius. An odd number of half twists result in a Möbius Band.

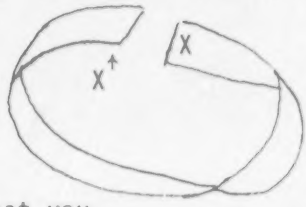
* A mathematicain confided
that a Möbius Band is one-sided,
and you'll get quite a laugh
if you cut one in half,
for it stays in one piece when divided.

(quoted in Bergamini, Mathematics)

TOPOLOGY

Investigation

1. Take a paper strip from the box. Place an "X" on both ends on the same side.



Make a loop with a half twist by glueing the ends together with the "X's" touching. You have made what mathematicians call a Möbius Band.* It has some interesting properties that you are going to investigate. To do so, you will need two more Möbius Bands; make them now like you did the first one.

2. Take one of your Bands. What do you think would happen if you cut the Band in half lengthwise? Would you have two loops? Try it. Carefully poke your scissors in the middle and cut around the loop in the lengthwise direction. What happened? If you cut it in half again, do you think the same thing would happen?

3. Take another of the Möbius Bands you made in step #1. What do you think would happen if you cut it in thirds? Begin cutting lengthwise one-third of the way from the edge. Continue cutting as long as you can, staying one-third from the edge. What result did you get?

4. Now try to color the third Möbius Band you made in step #1. Do you think you can color one side of the Band red and the other side blue? Try it. What happened? How many sides does a Möbius Band have? Can you color both sides of the loops you got in steps #2 and #3? Since a Möbius Band has only one side, coloring can be used as a test to see if a loop is a Möbius Band. If only one color can be used, it has only one side and is a Möbius Band.

5. Now take three more paper strips from the box and again mark both ends with "X's" on the same side. This time make loops with a full twist so when you glue the ends together, the "X's" do not touch. What do you think will happen when you cut one of these in half lengthwise? When you cut one one-third in from the edge? When you color one side red? Try it and see, using the cutting and coloring directions you followed in steps #2-4. Were you right?

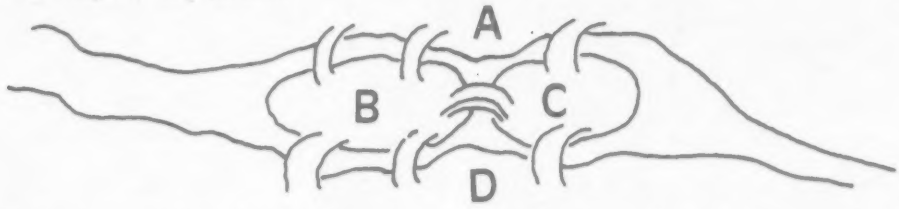
6. What do you think will happen with a loop made with 1½ twists ("X's" on ends touching when glued) when it is cut? When it is colored? What will happen with a loop made with two twists ("X's" on ends not touching)? What pattern do you see between Möbius Bands and half-twists in a loop?

* This little device was discovered by August Möbius, A German mathematician, in 1858.

TOPOLOGY

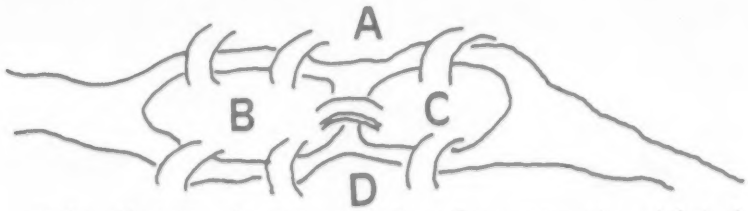
Worksheet

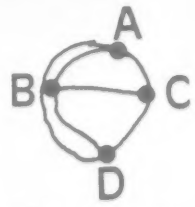
The properties of the Möbius Band are studies in a type of mathematics called topology which deals with properties of position such as "inside" or "between". It is said that topology began in 1736 when a famous mathematician, Euler, used ideas from this type of geometry to solve the problem of the Königsberg Bridges. The city of Königsberg in old Prussia was located on the banks and on two islands of the Pregel River. The parts of the town were connected by seven bridges, as shown:



On Sundays, the citizens would take a walk around the town. A popular question was: Is it possible to take a stroll around town, crossing each of the seven bridges just once and ending on the same side of the river where you began? Do you think it can be done? Try it.

To study the problem the map can be drawn in a simpler way by drawing each bridge as a curve connecting dots which represent the different parts of the town.





This diagram is an example of a network, which is a collection of points, called vertices, and lines connecting the vertices called arcs. In order to answer the Königsberg Bridge problem as Euler did, we will look further at network theory.

(1) For the network below, can you trace a path without lifting your pencil from the paper, tracing each arc once and only once? Can you trace such a path that ends where it began? Can you trace a path such that the beginning and ending points are not the same? Yes

By definition, an arc is a simple open curve connecting exactly 2 vertices.

(2) If a vertex has 2 arcs leading from it, can it be both the starting and ending point of a tracing path?* Is this true if it has any even number of arcs leading from it?* If a vertex has 3 arcs leading from it, can it be both the starting and ending point of a tracing path?**Is this true if it has any odd number of arcs leading from it? Yes

* Yes
** No

TOPOLOGY

Worksheet

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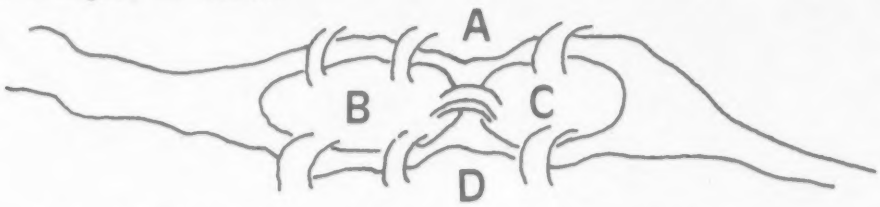

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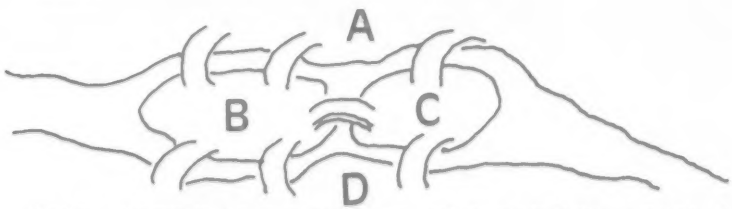

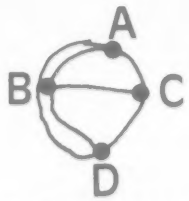

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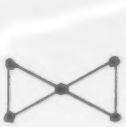
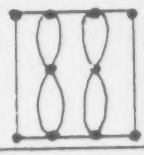
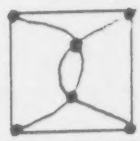
(2) If a vertex has 2 arcs leading from it, can it be both the starting and ending point of a tracing path? Is this true if it has any even number of arcs leading from it? If a vertex has 3 arcs leading from it, can it be both the starting and ending point of a tracing path? Is this true if it has any odd number of arcs leading from it?

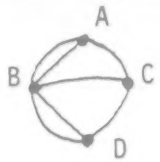
We will call vertices with an even number of arcs leading from them "even", and those with an odd number of arcs leading from them "odd". Can an even vertex be a starting and stopping point? Can an odd vertex?

(3) Try to trace each of these networks (without lifting your pencil and without retracting any arc) and complete the information in the chart:

Network						
Number of even vertices						
Number of odd vertices						
Traceable, ending at starting point?						
Traceable, not ending at starting point?						

(4) Study your completed chart. Do you see a pattern that you can use to predict whether or not a network can be traced before you try it? (If you need more examples, draw some networks to add to the chart.) If a network has no odd vertices, can it be traced both ways? If it has 2 odd vertices, can it be traced both ways? If it has more than 2 odd vertices, can it be traced? Why?

(5) Now back to the Königsberg Bridge problem:



How many odd vertices are there? Can it be traced ending at the starting point? Explain why not. Can you add one bridge in such a way that it is then traceable?

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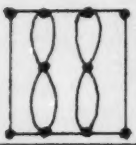
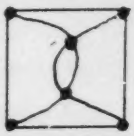
Network						
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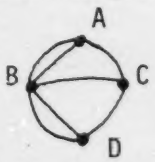


How many odd vertices are there? Can it be traced ending at the starting point? Explain why not. Can you add one bridge in such a way that it is then traceable?

TRIANGLES

Notes to the Instructor

Objectives:
- to discover that the triangle is the shape of a rigid frame and that other polygonal frames must be triangulated (made into triangles) in order to be rigid;
- to show that a triangle can have three sides, two sides, or no sides the same length;
- to show that the sum of the lengths of any two sides of a triangle must be greater than the length of the third side;
- to show that the sum of the angles of a triangle is 180° (of 2 right angles), and to use this with the fact that any polygon can be triangulated to find the sum of the interior angles of any polygon.

Materials:
- Strips cut from poster board with holes punched near the ends: 4" red strips, 5" green strips, 8" blue strips, 11" yellow strips, at least 6 of each per small group (3-4) students;
- Paper fasteners;
- Paper triangles: cut from unlined paper, several of various sizes and shapes per small group of students.

References:

Bruni, James V. and Silverman, Helene. "Using Geostrips and Anglefixers." The Arithmetic Teacher. (April 1975): 256-268.

Nuffield Mathematics Project: Mathematics: The Later Primary Years. New York: John Wiley & Sons, 1974: 99-100.

Comments: General

The concrete development of concepts in this Investigation can provide an important foundation for later more formal work in plane geometry. As with many of the Investigations, there can be definite advantages to having the students work in small groups of three or four. The tasks involved are time consuming and can be divided up among members of a group, each doing part, and then sharing the results. The dialogue which results with this technique adds to the understanding of the ideas involved.

The last part of the Investigation shows that the sum of the angles of a triangle fit along a straight line. Students may recognize this as being equal to 180° or as two right angles. For this part of the investigation, it is assumed that the students are familiar with the idea that angles are measured in degrees.

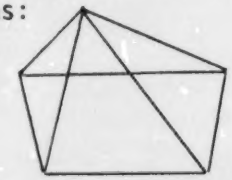
Comments: Specific for the Investigation

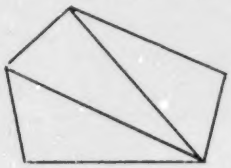
1. Small group work is beneficial here, both to save time in making all the different shapes, and for the discussion which will result among the students as they work. Note that a rectangle can be moved into a parallelogram and a square into a rhombus.

2. A variety of sizes and number of sides should be explored in order to lead to the generalization that only the triangle is rigid.

3. Adding a strip along the diagonal of the rectangle makes it rigid; the added strip is a brace. It is now rigid since it has been made into triangles, the only rigid shape. This is related to the real world situation of adding a brace to a sagging door.

4. Experimentation will result in various ways to make a frame rigid, such as:





The second way is more economical since it uses the fewest number of braces. The method for finding the way using the fewest number of braces is to begin at one "corner" and put in braces to each of the other non-adjacent "corners". When this is done, the region inside the frame is made into triangles; it is "triangulated". (See Worksheet #1.)

5. Triangles are important since they are the only rigid shape, and all other polygons can be built from triangles as indicated by triangulating them. Triangles of same-color strips are equilateral (have equal sides). All equilateral triangles are similar (have the same shape), but need not be congruent (the same size).

6. A triangle with two sides the same length (made with strips the same color) is isosceles.

7. A scalene triangle has no sides the same length (made with three different colored strips). Such triangles are only possible if the sum of the lengths of any two sides is greater than the length of the third side.

8. A triangle can have three equal sides (equilateral), two equal sides (isosceles), or no equal sides.

TRIANGLES

Investigation

1. Working with some friends, use the strips and fasteners to make a shape with three sides. Now take some more strips and make a shape with four sides and another with five sides. Which of these shapes wobble, that is, can be changed into different shapes? Is there any shape which does not wobble?

2. Undo the strips and make some different shapes with three sides, or four sides, or any number of sides. Test each shape to see if it can be moved into a different shape. What name can you give the shapes that are rigid and do not wobble?

3. Now take your four-sided shape and see if it wobbles. Can you make this shape rigid by adding a strip? What shapes do you notice inside the frame when you have made it right? If you had a screen door that sagged like this: , how would you use a board to make the door rigid?

4. Take the shapes with five sides, six sides, and seven sides. Try to make them rigid by adding the fewest possible number of strips to each shape. What shapes do you see inside each frame when you made it rigid? Are triangles formed? Can any straight sided shape (polygon) be made rigid this way? If you wanted a frame that would not wobble, what would you do? What does this say about the value of triangles in constructing things such as buildings?

5. Since triangles are so important, see what you can find out about them. First, investigate the sides of triangles. Can you make a triangle of all red strips? How is it like a triangle of all green strips? How is it different?

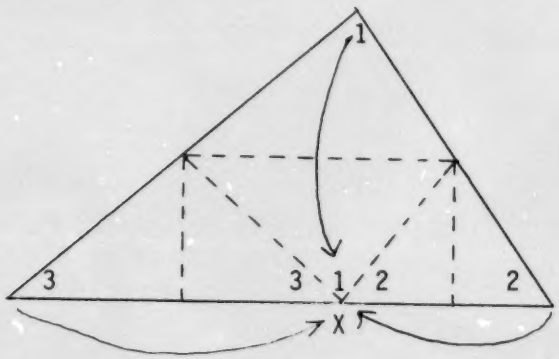
6. Can you make two different triangles each using just two colors of strips? How are they alike? How are they different?

7. Can you make a triangle using three different colored strips? Can you do this with some of the colors but not others? Why can you make a red-green-blue-triangle but not a red-green-yellow triangle? What must be true of the strips in order for them to make a triangle?

8. What have you found about the sides of a triangle? Can the sides all be the same length? Can two sides be the same length? Can all three sides be different in length?

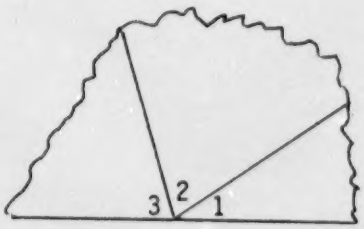
9. Given three different lengths, only one triangle is possible, but given four different lengths, six non-congruent quadrilaterals are possible. This emphasizes that the shape and size of a triangle are completely determined (it is rigid) by its sides.

10. Repeating this investigation with triangles of obviously different sized angles may be necessary in order for the students to be convinced that the angles of <u>any</u> triangle can be made to fit along a straight line. This shows the measure of the sum of the angles of a triangle is equal to that of a straight line: 180^{0} or 2 right angles. Some students may not realize that tearing the angles off the triangles does not change their measure. In this case, another method of demonstrating this sum is equal to a straight line may be used. Another suggestion: fold a cut out triangle as shown, and fit the angles together at point "X".



9. Take three different colored strips that do make a triangle. How many triangles of different size and shape (non-congruent) can you make with the three strips as sides? Take four different colored strips and make a four-sided figure (quadrilateral). How many non-congruent quadrilaterals can you make with those same strips? What has this to do with the rigidity of triangles and the non-rigidity of quadrilaterals?

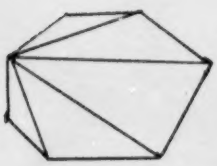
10. Now investigate the corners (angles) of a triangle. Take a paper triangle and label the corners 1, 2, and 3. Tear off the corners, and try to fit them together (side to side) as shown below. Do they fit on a straight line? Will this work with any triangle, no matter what shape it is? Try one of a different size and shape if you are not sure. What can you say about the sum of the angles of any triangle?



TRIANGLES

Worksheet #1

1. You have found that the most economical way to get a rigid polygonal frame was to make triangles inside the shape by adding strips, starting at one corner and going to each of the other corners where possible. Do you think any polygonal frame can be turned into triangles (triangulated) in this way? Complete the chart below by drawing in the strips added to make it rigid.

Frame	Number of sides of the frame	Minimum number of strips added to make it rigid	Number of triangles formed
	3	0	1
	4	1	2
	5	2	3
	6	3	4
	7	4	5
	8	5	6
Generalization	n	n-3	n-2

TRIANGLES

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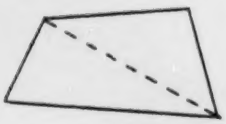
	4	1	2
	5		
	6		
	7		
	8		

2. Try to find number patterns in your table.

   How does the number of sides compare to the number of strips added?

   Number of sides - 3 = number of added strips

   How does the number of sides compare to the number of triangles formed?

   Number of sides - 2 = number of triangles formed

3. Use these patterns to complete the following:

   A polygonal frame with 10 sides needs 7 strips to make it into 8 triangles.

   A polygonal frame with 25 sides needs 22 strips to make it into 23 triangles.

   A polygonal frame with 100 sides needs 97 strips to make it into 98 triangles.

2. Try to find number patterns in your table.

   How does the number of sides compare to the number of strips added?

   _____

   How does the number of sides compare to the number of triangles formed?

   _____

3. Use these patterns to complete the following:

   A polygonal frame with 10 sides needs _____ strips to make it into _____ triangles.

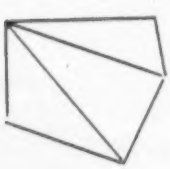
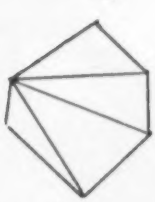
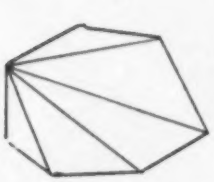
   A polygonal frame with 25 sides needs _____ strips to make it into _____ triangles.

   A polygonal frame with 100 sides needs _____ strips to make it into _____ triangles.

TRIANGLES

Worksheet #2

1. You have found that the sum of the angles of a triangle is equal to a straight line or 180^0, and that any polygonal frame can be made into triangles. How does this help you find the sum of the angles of any polygonal frame? Complete the chart by first drawing in the lines needed to divide the shape into triangles, then answer the questions using the triangles formed.

Shape	Number of sides	Number of triangles formed	Sum of angles in each triangle	Sum of angles of shape
	3	1	180°	180°
	4	2	180°	360°
	5	3	180°	540°
	6	4	180°	720°
	7	5	180°	900°

2. Can you find a short way to complete the following without drawing?
A shape with 10 sides can be made into 8 triangles; the sum of its angles is $8 \times 180^0 = 1440^0$.

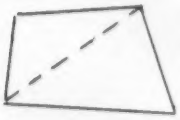
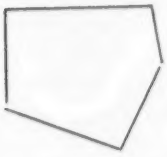
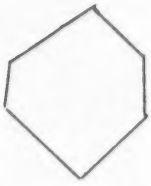
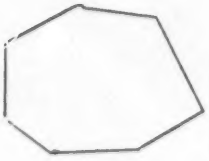
A shape with 25 sides can be made into 23 triangles; the sum of its angles is $23 \times 180^0 = 4140^0$.

A shape with 100 sides can be made into 98 triangles; the sum of its angles is $98 \times 180^0 = 17640^0$.

TRIANGLES

Worksheet #2

1. You have found that the sum of the angles of a triangle is equal to a straight line or 180^0, and that any polygonal frame can be made into triangles. How does this help you find the sum of the angles of any polygonal frame? Complete the chart by first drawing in the lines needed to divide the shape into triangles, then answer the questions using the triangles formed.

Shape	Number of sides	Number of triangles formed	Sum of angles in each triangle	Sum of angles of shape
	3	1	180°	180°
	4	2	180°	360°
				
				
				

2. Can you find a short way to complete the following without drawing?
   A shape with 10 sides can be made into _____ triangles; the sum of its angles is _____.

   A shape with 25 sides can be made into _____ triangles; the sum of its angles is _____.

   A shape with 100 sides can be made into _____ triangles; the sum of its angles is _____.

Worksheet #3

Where are triangles used?

1. Look around the room. Where do you see triangles?

_____

_____

2. What other places can you think of seeing triangles used? Why are they used in these places?

_____

_____

3. How can you use what you have found out about triangles to help explain why the Egyptian Pyramids are still standing while other buildings have fallen?

_____

_____

4. Why do you think triangles are studied so much in geometry?

_____

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_____

The following were members of the Guidelines Committee for Quality Mathematics Teaching during the development of this monograph:

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